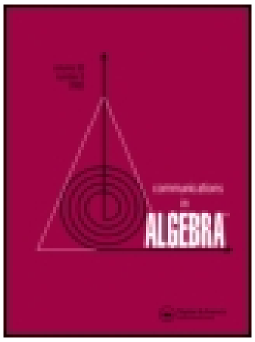

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

Finite basis for radical well-mixed difference ideals generated by binomials

Jie Wang

COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2017.1392541

Finite basis for radical well-mixed difference ideals generated by binomials

Jie Wang

School of Mathematical Sciences, Peking University, Beijing, China

ABSTRACT
In this paper, we prove a finite basis theorem for radical well-mixed difference ideals generated by binomials. As a consequence, every strictly ascending chain of radical well-mixed difference ideals generated by binomials in a difference polynomial ring is finite, which solves an open problem in difference algebra raised by Hrushovski in the binomial case.

ARTICLE HISTORY
Received 17 May 2017
Communicated by J. Bell

KEYWORDS
Binomial difference ideal; finitely generated property; Hrushovski's problem; well-mixed difference ideal

2010 MATHEMATICS SUBJECT CLASSIFICATION
12H10

1. Introduction

In [4], Hrushovski developed the theory of difference schemes, which is one of the major recent advances in difference algebra geometry. In Hrushovski's theory, well-mixed difference ideals played a key role. Therefore, it is significant to make clear of the properties of well-mixed difference ideals.

It is well-known that Hilbert's basis theorem does not hold for difference ideals in a difference polynomial ring. Instead, we have Ritt–Raudenbush basis theorem which asserts that every perfect difference ideal in a difference polynomial ring has a finite basis. It is naturally to ask if the finitely generated property holds for more difference ideals. Let K be a difference field and R a finitely difference generated difference algebra over K. In [4, Section 4.6], Hrushovski raised the problem whether a radical well-mixed difference ideal in R is finitely generated. The problem is also equivalent to whether the ascending chain condition holds for radical well-mixed difference ideals in R. For the sake of convenience, let us state it as a conjecture:

Conjecture 1.1. *Every strictly ascending chain of radical well-mixed difference ideals in R is finite.*

Also in [4, Section 4.6], Hrushovski proved that the answer is yes under some additional assumptions on R. In [5], Levin showed that the ascending chain condition does not hold if we drop the radical condition. The counterexample given by Levin is a well-mixed difference ideal generated by binomials. In [9, Section 9], Wibmer showed that if R can be equipped with the structure of a difference Hopf algebra over K, then Conjecture 1.1 is valid. In [7], Wang proved that Conjecture 1.1 is valid for radical well-mixed difference ideals generated by monomials.

Difference ideals generated by binomials were first studied by Gao et al. [3]. Some basic properties of difference ideals generated by binomials were proved in that paper due to the correspondence between $\mathbb{Z}[x]$-lattices and normal binomial difference ideals.

The main result of this paper is that every radical well-mixed difference ideal generated by binomials in a difference polynomial ring over an algebraic closed and inversive difference field is finitely generated.

CONTACT Jie Wang wangjie212@mails.ucas.ac.cn School of Mathematical Sciences, Peking University, Beijing, China.

As a consequence, Conjucture 1.1 is valid for radical well-mixed difference ideals generated by binomials in a difference polynomial ring over an algebraic closed and inversive difference field.

2. Preliminaries

2.1. Preliminaries for difference algebra

We recall some basic notions from difference algebra. Standard references are [5, 8]. All rings in this paper will be assumed to be commutative and unital.

A *difference ring*, or *σ-ring* for short, is a ring R together with a ring endomorphism $\sigma : R \to R$, and we call σ a *difference operator* on R. If R is a field, then we call it a *difference field*, or *σ-field* for short. A typical example of σ-field is the field of rational functions $\mathbb{Q}(x)$ with $\sigma(f(x)) = f(x+1)$. In this paper, all σ-fields will be assumed to be of characteristic 0.

Following Gao et al. [2], we introduce the following notation of symbolic exponents. Let x be an algebraic indeterminate and $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{N}[x]$. For a in a σ-ring, we denote $a^p = \prod_{i=0}^{s}(\sigma^i(a))^{c_i}$ with $\sigma^0(a) = a$ and $a^0 = 1$. It is easy to check that for $p, q \in \mathbb{N}[x]$, we have $a^{p+q} = a^p a^q, a^{pq} = (a^p)^q$.

Let R be a σ-ring. A *σ-ideal* I in R is an algebraic ideal which is closed under σ, i.e., $\sigma(I) \subseteq I$. If I also has the property that $a^x \in I$ implies $a \in I$, it is called a *reflexive σ-ideal*. A *σ-prime* σ-ideal is a reflexive σ-ideal which is prime as an algebraic ideal. A σ-ideal I is said to be *well-mixed* if for $a, b \in R$, $ab \in I$ implies $ab^x \in I$. A σ-ideal I is said to be *perfect* if for $a \in R$ and $g \in \mathbb{N}[x] \setminus \{0\}$, $a^g \in I$ implies $a \in I$. It is easy to prove that every perfect σ-ideal is well-mixed and every σ-prime σ-ideal is perfect.

If $F \subseteq R$ is a subset of R, then we denote the minimal ideal containing F by (F), the minimal σ-ideal containing F by $[F]$ and denote the minimal well-mixed σ-ideal, the minimal radical well-mixed σ-ideal, the minimal perfect σ-ideal containing F by $\langle F\rangle$, $\langle F\rangle_r$, $\{F\}$, respectively, which are called the *well-mixed closure*, the *radical well-mixed closure*, the *perfect closure* of F, respectively.

Let K be a σ-field and $\mathbb{Y} = (y_1, \dots, y_n)$ a tuple of σ-indeterminates over K. Then the *σ-polynomial ring* over K in $\mathbb{Y}$ is the polynomial ring in the variables $y_i^{x^j}$ for $i = 1, \dots, n$ and $j \in \mathbb{N}$. It is denoted by $K\{\mathbb{Y}\} = K\{y_1, \dots, y_n\}$ and has a natural K-σ-algebra structure.

2.2. Preliminaries for binomial difference ideals

A $\mathbb{Z}[x]$-*lattice* is a $\mathbb{Z}[x]$-submodule of $\mathbb{Z}[x]^n$ for some n. Since $\mathbb{Z}[x]^n$ is Noetherian as a $\mathbb{Z}[x]$-module, we see that any $\mathbb{Z}[x]$-lattice is finitely generated as a $\mathbb{Z}[x]$-module. If $\mathbf{f}_1, \dots, \mathbf{f}_m$ generates a $\mathbb{Z}[x]$-lattice L, then we write $L = (\mathbf{f}_1, \dots, \mathbf{f}_m)$.

Let K be a σ-field and $\mathbb{Y} = (y_1, \dots, y_n)$ a tuple of σ-indeterminates over K. For $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{N}[x]^n$, we define $\mathbb{Y}^{\mathbf{f}} = \prod_{i=1}^{n} y_i^{f_i}$. $\mathbb{Y}^{\mathbf{f}}$ is called a *monomial* in $\mathbb{Y}$ and $\mathbf{f}$ is called its *support*. For $a, b \in K^* = K \setminus \{0\}$ and $\mathbf{f}, \mathbf{g} \in \mathbb{N}[x]^n$, $a\mathbb{Y}^{\mathbf{f}} + b\mathbb{Y}^{\mathbf{g}}$ is called a *binomial*. If $a = 1, b = -1$, then $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}}$ is called a *pure binomial*. A *(pure) binomial σ-ideal* is a σ-ideal generated by (pure) binomials.

For $f \in \mathbb{Z}[x]$, we write $f = f_+ - f_-$, where $f_+, f_- \in \mathbb{N}[x]$ are the positive part and the negative part of f, respectively. For $\mathbf{f} \in \mathbb{Z}[x]^n$, $\mathbf{f}_+ = (f_{1+}, \dots, f_{n+})$, $\mathbf{f}_- = (f_{1-}, \dots, f_{n-})$.

Definition 2.1. A *partial character* ρ on a $\mathbb{Z}[x]$-lattice L is a group homomorphism from L to the multiplicative group K^* satisfying $\rho(x\mathbf{f}) = (\rho(\mathbf{f}))^x$ for all $\mathbf{f} \in L$.

A *trivial* partial character on a $\mathbb{Z}[x]$-lattice L is defined by setting $\rho(\mathbf{f}) = 1$ for all $\mathbf{f} \in L$.

Given a partial character ρ on a $\mathbb{Z}[x]$-lattice L, we define the following binomial σ-ideal in $K\{\mathbb{Y}\}$,

$$\mathcal{I}_L(\rho) := [\mathbb{Y}^{\mathbf{f}_+} - \rho(\mathbf{f})\mathbb{Y}^{\mathbf{f}_-} \mid \mathbf{f} \in L].$$

L is called the *support lattice* of $\mathcal{I}_L(\rho)$. In particular, if ρ is a trivial partial character on L, then the binomial σ-ideal defined by ρ is called a *lattice σ-ideal*, which is denoted by $\mathcal{I}_L$.

Let $\mathbb{m}$ be the multiplicatively closed set generated by $y_i^{x^j}$ for $i = 1, \ldots, n, j \in \mathbb{N}$. A σ-ideal I is said to be *normal* if for any $M \in \mathbb{m}$ and $p \in K\{\mathbb{Y}\}$, $Mp \in I$ implies $p \in I$. For any σ-ideal I,

$$I : \mathbb{m} = \{p \in K\{\mathbb{Y}\} \mid \exists M \in \mathbb{m} \text{s.t.} Mp \in I\}$$

is a normal σ-ideal.

Lemma 2.2 ([3, Corollary 6.20]). *A normal binomial σ-ideal is radical.*

Proof. For the proof, please refer to Gao et al. [3]. □

In [3, Theorem 6.19], it was proved that there is a one-to-one correspondence between normal binomial σ-ideals and partial characters ρ on some $\mathbb{Z}[x]$-lattice L.

In [3], the concept of *M-saturation* of a $\mathbb{Z}[x]$-lattice was introduced.

Definition 2.3. Assume that K is algebraically closed. If a $\mathbb{Z}[x]$-lattice L satisfies

$$m\mathbf{f} \in L \Rightarrow (x - o_m)\mathbf{f} \in L, \tag{1}$$

where $m \in \mathbb{N}$, $\mathbf{f} \in \mathbb{Z}[x]^n$, and o_m is the m-th transforming degree of the unity of K (see [3, Lemma 5.13] for the definition), then it is said to be *M-saturated*. For any $\mathbb{Z}[x]$-lattice L, the smallest M-saturated $\mathbb{Z}[x]$-lattice containing L is called the *M-saturation* of L and is denoted by $\text{sat}_M(L)$.

The following two lemmas were proved in [3] for the Laurent case and it is easy to generalize to the normal case.

Lemma 2.4 ([3, Theorem 5.21]). *Assume that K is algebraically closed and inversive. Let ρ be a partial character on a $\mathbb{Z}[x]$-lattice L. If $\mathcal{I}_L(\rho)$ is well mixed, then L is M-saturated. Conversely, if L is M-saturated, then either $\langle \mathcal{I}_L(\rho)\rangle : \mathbb{m} = [1]$ or $\mathcal{I}_L(\rho)$ is well-mixed.*

Lemma 2.5 ([3, Theorem 5.23]). *Assume that K is algebraically closed and inversive. Let ρ be a partial character on a $\mathbb{Z}[x]$-lattice L. Then $\langle \mathcal{I}_L(\rho)\rangle_r : \mathbb{m}$ is either $[1]$ or a normal binomial σ-ideal whose support lattice is $\text{sat}_M(L)$. In particular, $\langle \mathcal{I}_L\rangle_r : \mathbb{m}$ is either $[1]$ or $\mathcal{I}_{\text{sat}_M(L)}$.*

3. Radical well-mixed difference ideal generated by binomials is finitely generated

In this section, we will prove that every radical well-mixed σ-ideal generated by binomials in a σ-polynomial ring over an algebraic closed and inversive σ-field is finitely generated as a radical well-mixed σ-ideal. For simplicity, we only consider the case for pure binomials since it is easy to generalize to the general case.

For convenience, for $h \in \mathbb{Z}[x]$, if $\deg(h_+) > \deg(h_-)$, then we set $h^+ = h_+$ and $h^- = h_-$. Otherwise, we set $h^+ = h_-$ and $h^- = h_+$. Moveover, we set $\deg(0) = -1$.

For $a, b, c, d \in \mathbb{N}$, we define $ax^b > cx^d$ if $b > d$, or $b = d$ and $a > c$. For $h \in \mathbb{Z}[x]$, we use $\text{lt}(h)$ and $\text{lc}(h)$ to denote the leading term and the leading coefficient of h respectively.

Theorem 3.1. *For any $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^n$, $\langle \mathcal{I}_L\rangle_r$ is finitely generated as a radical well-mixed σ-ideal.*

Proof. Denote the set of all maps from $\{1, \ldots, n\}$ to $\{+, -, 0\}$ by Λ and $\tau_0 \in \Lambda$ is the map such that $\tau_0(i) = 0$ for $1 \le i \le n$. Let $\Lambda_0 = \Lambda \backslash \{\tau_0\}$. For any $\tau \in \Lambda_0$, we define

$$A_\tau := \{(h_1, \ldots, h_n) \in L \mid \text{lc}(h_i) > 0 \text{ if } \tau(i) = +,\ \text{lc}(h_i) < 0 \text{ if } \tau(i) = -, \text{ and } \text{lc}(h_i) = 0 \text{ if } \tau(i) = 0,\ i = 1, \ldots, n\},$$

and

$$\Sigma_\tau := \{(\deg(h_1^+), \mathrm{lc}(h_1^+), \ldots, \deg(h_n^+), \mathrm{lc}(h_n^+), \deg(h_1^-), \ldots, \deg(h_n^-)) \mid (h_1, \ldots, h_n) \in A_\tau\}.$$

For any $\tau \in \Lambda_0$, let G_τ be the subset of A_τ such that

$$\{(\deg(g_1^+), \mathrm{lc}(g_1^+), \ldots, \deg(g_n^+), \mathrm{lc}(g_n^+), \deg(g_1^-), \ldots, \deg(g_n^-)) \mid \mathbf{g} = (g_1, \ldots, g_n) \in G_\tau\}$$

is the set of minimal elements in Σ_τ under the product order. It is clear that G_τ is a finite set. Let

$$F_\tau := \{\mathbb{Y}^{\mathbf{g}_+} - \mathbb{Y}^{\mathbf{g}_-} \mid \mathbf{g} \in G_\tau\}.$$

We claim that the finite set $\cup_{\tau\in\Lambda_0} F_\tau$ generates $\langle \mathcal{I}_L \rangle_r$ as a radical well-mixed σ-ideal.

Let $\mathcal{I}_0 = \langle \cup_{\tau\in\Lambda_0} F_\tau \rangle_r$. We will prove the claim by showing that $\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-} \in \mathcal{I}_0$ for all $\mathbf{h} \in L$. Let us do induction on $(\mathrm{lt}(h_1^+), \ldots, \mathrm{lt}(h_n^+))$ under the product order for $\mathbf{h} = (h_1, \ldots, h_n) \in L$. For simplicity, we will assume that $\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}$ has the form

$$y_1^{h_1^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} - y_1^{h_1^-} \cdots y_t^{h_t^-} y_{t+1}^{h_{t+1}^+} \cdots y_n^{h_n^+},$$

where $1 \leq t \leq n$. And without loss of generality, we further assume $\mathrm{lc}(h_i) \neq 0$ for $1 \leq i \leq n$.

The case for $\mathbf{h} = \mathbf{0}$ is trivial. Now for the inductive step. By definition, there exists $\tau \in \Lambda_0$ and $(g_1, \ldots, g_n) \in G_\tau$ such that $(h_1, \ldots, h_n) \in A_\tau$ and $\deg(g_i^+) \leq \deg(h_i^+), \mathrm{lc}(g_i^+) \leq \mathrm{lc}(h_i^+), \deg(g_i^-) \leq \deg(h_i^-), i = 1, \ldots, n$. Let us choose a $j \in \{1, \ldots, n\}$ such that

$$\deg(h_j^+) - \deg(g_j^+) = \min_{1\leq i\leq n} \{\deg(h_i^+) - \deg(g_i^+)\}.$$

Without loss of generality, we can assume $j = 1$. Let $s = \deg(h_1^+) - \deg(g_1^+) \geq 0$. Since $\mathrm{lc}(h_1^+) \geq \mathrm{lc}(g_1^+)$, there exists an $e \in \mathbb{N}[x]$ such that $\deg(e) < \deg(h_1^+)$ and $p = h_1^+ + e - x^s g_1^+ \in \mathbb{N}[x]$ with $\mathrm{lt}(p) < \mathrm{lt}(h_1^+)$. Then

$$\begin{aligned}
& y_1^e y_2^{x^s g_2^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (y_1^{h_1^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} - y_1^{h_1^-} \cdots y_t^{h_t^-} y_{t+1}^{h_{t+1}^+} \cdots y_n^{h_n^+}) \\
&= y_1^{p+x^s g_1^+} y_2^{h_2^+ + x^s g_2^+} \cdots y_t^{h_t^+ + x^s g_t^+} y_{t+1}^{h_{t+1}^- + x^s g_{t+1}^-} \cdots y_n^{h_n^- + x^s g_n^-} \\
&\quad - y_1^{h_1^- + e} y_2^{h_2^- + x^s g_2^+} \cdots y_t^{h_t^- + x^s g_t^+} y_{t+1}^{h_{t+1}^+ + x^s g_{t+1}^-} \cdots y_n^{h_n^+ + x^s g_n^-} \\
&= (y_1^{g_1^+} \cdots y_t^{g_t^+} y_{t+1}^{g_{t+1}^-} \cdots y_n^{g_n^-} - y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{g_{t+1}^+} \cdots y_n^{g_n^+})^{x^s} y_1^p y_2^{h_2^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} \\
&\quad + y_1^{p+x^s g_1^-} y_2^{h_2^+ + x^s g_2^-} \cdots y_t^{h_t^+ + x^s g_t^-} y_{t+1}^{h_{t+1}^- + x^s g_{t+1}^+} \cdots y_n^{h_n^- + x^s g_n^+} \\
&\quad - y_1^{h_1^- + e} y_2^{h_2^- + x^s g_2^+} \cdots y_t^{h_t^- + x^s g_t^+} y_{t+1}^{h_{t+1}^+ + x^s g_{t+1}^-} \cdots y_n^{h_n^+ + x^s g_n^-} \\
&= (y_1^{g_1^+} \cdots y_t^{g_t^+} y_{t+1}^{g_{t+1}^-} \cdots y_n^{g_n^-} - y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{g_{t+1}^+} \cdots y_n^{g_n^+})^{x^s} y_1^p y_2^{h_2^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} \\
&\quad + y_1^{d_1} \cdots y_n^{d_n} (y_1^{w_{1+}} \cdots y_n^{w_{n+}} - y_1^{w_{1-}} \cdots y_n^{w_{n-}}),
\end{aligned}$$

for some $d_1, \ldots, d_n \in \mathbb{N}[x]$ and some $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}[x]^n$. It is clear that $\mathbf{w} \in L$. Since $\mathrm{lt}(p + x^s g_1^-) < \mathrm{lt}(h_1^+), \mathrm{lt}(h_1^- + e) < \mathrm{lt}(h_1^+)$, then $\mathrm{lt}(w_1^+) < \mathrm{lt}(h_1^+)$, and because of the choice of j, we have $s + \deg(g_i^+) \leq \deg(h_i^+)$ for $2 \leq i \leq n$, from which it follows $\mathrm{lt}(w_i^+) \leq \mathrm{lt}(h_i^+), 2 \leq i \leq n$. Therefore, $(\mathrm{lt}(w_1^+), \ldots, \mathrm{lt}(w_n^+)) < (\mathrm{lt}(h_1^+), \ldots, \mathrm{lt}(h_n^+))$. Thus by the induction hypothesis, $y_1^{w_{1+}} \cdots y_n^{w_{n+}} - y_1^{w_{1-}} \cdots y_n^{w_{n-}} \in \mathcal{I}_0$ and hence

$$y_1^e y_2^{x^s g_2^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0.$$

So by the properties of radical well-mixed σ-ideals, we have

$$y_1^{x^s g_1^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0,$$

and then

$$y_1^{x^s g_1^-} \cdots y_t^{x^s g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0.$$

If $s > 0$, let $s' = \max\{0, s - \min_{1 \le i \le t}\{\deg(g_i^+) - \deg(g_i^-)\}\} < s$. Again by the properties of radical well-mixed σ-ideals, we have

$$y_1^{x^{s'} g_1^+} \cdots y_t^{x^{s'} g_t^+} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0,$$

and then

$$y_1^{x^{s'} g_1^-} \cdots y_t^{x^{s'} g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0.$$

If $s' > 0$, repeat the above process, and we eventually obtain

$$y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0.$$

Since $\deg(g_i^-) \le \deg(h_i^-)$, $1 \le i \le t$ and $s + \deg(g_i^+) \le \deg(h_i^+)$, $t+1 \le i \le n$, then by the properties of radical well-mixed σ-ideals, we have

$$\mathbb{Y}^{\mathbf{h}_-}(\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0. \tag{2}$$

Similarly, we also have

$$\mathbb{Y}^{\mathbf{h}_+}(\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-}) \in \mathcal{I}_0. \tag{3}$$

Combining (2) and (3), we obtain $(\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-})^2 \in \mathcal{I}_0$, and hence $\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-} \in \mathcal{I}_0$. So we complete the proof. □

Corollary 3.2. *Let $L \subseteq \mathbb{Z}[x]^n$ be a $\mathbb{Z}[x]$-lattice such that $\mathcal{I}_L$ is well-mixed, then $\mathcal{I}_L$ is finitely generated as a radical well-mixed σ-ideal.*

Proof. It is immediate from Theorem 3.1 since $\mathcal{I}_L$ is already a radical well-mixed σ-ideal. □

Example 3.3. Let $L = (\begin{pmatrix} x-1 \\ 1-x \end{pmatrix}) \subseteq \mathbb{Z}[x]^2$ be a $\mathbb{Z}[x]$-lattice. Since L is saturated, $\mathcal{I}_L$ is a σ-prime σ-ideal [3, Corollary 6.22(c)] and hence well mixed. Then by Theorem 3.1, $\mathcal{I}_L = [y_1^{x^i} y_2 - y_1 y_2^{x^i} : i \in \mathbb{N}^*] = \langle y_1^x y_2 - y_1 y_2^x \rangle_r$.

Example 3.4. Let $L = (\begin{pmatrix} x^2+1-x \\ x-1 \end{pmatrix}) \subseteq \mathbb{Z}[x]^2$ be a $\mathbb{Z}[x]$-lattice. Since L is saturated, $\mathcal{I}_L$ is a σ-prime σ-ideal and hence well-mixed. Then by Theorem 3.1, $\mathcal{I}_L = \langle y_1^{x^2+1} y_2^x - y_1^x y_2, y_1^{x^3+1} y_2^{x^2} - y_2 \rangle_r$.

To show that radical well-mixed σ-ideals generated by binomials are finitely generated, we need the following lemma.

Lemma 3.5 ([7, Proposition 5.2])**.** *Let F and G be subsets of any σ-ring R. Then*

$$\langle F \rangle_r \cap \langle G \rangle_r = \langle FG \rangle_r.$$

As a corollary, if I and J are two σ-ideals of R, then

$$\langle I \rangle_r \cap \langle J \rangle_r = \langle I \cap J \rangle_r = \langle IJ \rangle_r.$$

Proof. For the proof, please refer to Wang [7]. □

Lemma 3.6. *Assume that K is algebraically closed and inversive. Suppose that $I \subseteq K\{\mathbb{Y}\}$ is a pure binomial σ-ideal. Then $\langle I\rangle_r : \mathbb{m}$ is finitely generated as a radical well-mixed σ-ideal.*

Proof. Since $I : \mathbb{m}$ is a normal binomial σ-ideal, there exists a $\mathbb{Z}[x]$-lattice L such that $I : \mathbb{m} = \mathcal{I}_L$. Note that $\langle I\rangle_r : \mathbb{m}=\langle I : \mathbb{m}\rangle_r : \mathbb{m}$, so by Lemma 2.5, $\langle I\rangle_r : \mathbb{m}$ is [1] or $\mathcal{I}_{\mathrm{sat}_M(L)}$. Since $\langle I\rangle_r$ is radical well mixed, it is easy to show that $\langle I\rangle_r : \mathbb{m}$ is also radical well mixed. So by Corollary 3.2, $\langle I\rangle_r : \mathbb{m}$ is finitely generated as a radical well-mixed σ-ideal. □

Lemma 3.7. *Assume that K is algebraically closed and inversive. Suppose that $I \subseteq K\{\mathbb{Y}\}$ is a pure binomial σ-ideal. Then*

$$\langle I\rangle_r = \langle I\rangle_r : \mathbb{m} \cap \langle I, y_{p_1}^{x^{a_1}}\rangle_r \cap \cdots \cap \langle I, y_{p_l}^{x^{a_l}}\rangle_r$$

for some $\{p_1,\dots,p_l\} \subseteq \{1,\dots,n\}$ and some $(a_1,\dots,a_l) \in \mathbb{N}^l$.

Proof. By Lemma 3.6, $\langle I\rangle_r : \mathbb{m}$ is finitely generated as a radical well-mixed σ-ideal. Therefore, there exist $f_1,\dots,f_s \in \langle I\rangle_r : \mathbb{m}$ and $m_1,\dots,m_s \in \mathbb{m}$ such that $\langle I\rangle_r : \mathbb{m} = \langle f_1,\dots,f_s\rangle_r$ and $m_1f_1,\dots,m_sf_s \in \langle I\rangle_r$. Then by Lemma 3.5,

$$\begin{aligned}
\langle I\rangle_r &= \langle I, f_1\rangle_r \cap \langle I, m_1\rangle_r\\
&= \langle I, f_1, f_2\rangle_r \cap \langle I, f_1, m_2\rangle_r \cap \langle I, m_1\rangle_r\\
&= \langle I, f_1, f_2\rangle_r \cap \langle I, m_1m_2\rangle_r\\
&= \cdots\\
&= \langle f_1,\dots,f_s\rangle_r \cap \langle I, m_1\cdots m_s\rangle_r\\
&= \langle I\rangle_r : \mathbb{m} \cap \langle I, y_{p_1}^{x^{a_1}}\rangle_r \cap \cdots \cap \langle I, y_{p_l}^{x^{a_l}}\rangle_r,
\end{aligned}$$

for some $\{p_1,\dots,p_l\} \subseteq \{1,\dots,n\}$ and some $(a_1,\dots,a_l) \in \mathbb{N}^l$. □

Suppose that $\{j_1,\dots,j_t\} \subseteq \{1,\dots,n\}$, $(a_1,\dots,a_t) \in \mathbb{N}^t$ and $I_0 \subseteq K\{y_1,\dots,y_n\}$ is a pure binomial σ-ideal. Let $T_{j_1\dots j_t}^{a_1\dots a_t} = \{y_1^{f_1}\cdots y_n^{f_n} \mid f_1,\dots,f_n \in \mathbb{N}[x], \deg(f_{j_i}) < a_i, 1 \le i \le t\}$. We say that I_0 is *saturated* with respect to $\{y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\}$ if $I_0 = I_0 : T_{j_1\dots j_t}^{a_1\dots a_t}$, that is, for any $g \in K\{y_1,\dots,y_n\}$ and $M \in T_{j_1\dots j_t}^{a_1\dots a_t}$, $Mg \in I_0$ implies $g \in I_0$. Let $I \subseteq K\{y_1,\dots,y_n\}$ be a pure binomial σ-ideal. The minimal σ-ideal containing I which is saturated with respect to $\{y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\}$ is called the *$T_{j_1\dots j_t}^{a_1\dots a_t}$-saturated closure* of I, denoted by $N_{j_1\dots j_t}^{a_1\dots a_t}(I)$. We will give a concrete description of the $T_{j_1\dots j_t}^{a_1\dots a_t}$-saturated closure of a pure binomial σ-ideal I. Let $I^{[0]} = I$ and recursively define $I^{[i]} = [I^{[i-1]} : T_{j_1\dots j_t}^{a_1\dots a_t}](i = 1,2,\dots)$. The following lemma is easy to check by definition.

Lemma 3.8. *Let $I \subseteq K\{y_1,\dots,y_n\}$ be a pure binomial σ-ideal. Then*

$$N_{j_1\dots j_t}^{a_1\dots a_t}(I) = \cup_{i=0}^{\infty} I^{[i]}. \tag{4}$$

Let $I_0 \subseteq K\{y_1,\dots,y_n\}$ be a pure binomial σ-ideal. Then we say $I = \langle I_0, y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r$ is *quasi-normal* if I_0 is saturated with respect to $\{y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\}$ and for any binomial $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}} \in I_0$, if $\mathbb{Y}^{\mathbf{f}} \in [y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$, then $\mathbb{Y}^{\mathbf{g}} \in [y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$. In analogy with Theorem 3.1, we can prove the following lemma.

Lemma 3.9. *Let $\{j_1,\dots,j_t\} \subseteq \{1,\dots,n\}$, $(a_1,\dots,a_t) \in \mathbb{N}^t$ and $I_0 \subseteq K\{y_1,\dots,y_n\}$ a pure binomial σ-ideal. Assume that $I = \langle I_0, y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r$ is quasi-normal. Then I is finitely generated as a radical well-mixed σ-ideal.*

Proof. Let $J = \{\mathbb{Y}^{\mathbf{h}_+} - \mathbb{Y}^{\mathbf{h}_-} \in I_0 \mid \mathbb{Y}^{\mathbf{h}_+}, \mathbb{Y}^{\mathbf{h}_-} \in T_{j_1\ldots j_t}^{a_1\ldots a_t}\}$. By a similar argument with Theorem 3.1, we can prove $\langle J\rangle_r$ is finitely generated as a radical well-mixed σ-ideal. It follows that $I = \langle J, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r$ is finitely generated as a radical well-mixed σ-ideal. □

Lemma 3.10. *Suppose that* $\{j_1,\ldots,j_t\} \subseteq \{1,\ldots,n\}$, $(a_1,\ldots,a_t) \in \mathbb{N}^t$ *and* $I \subseteq K\{\mathbb{Y}\}$ *is a pure binomial* σ*-ideal. Let* $I_0 = N_{j_1\ldots j_t}^{a_1\ldots a_t}(I)$. *Assume that* $\langle I_0, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r$ *is quasi-normal. Then there exist* $\{p_1,\ldots,p_l\} \subseteq \{1,\ldots,n\}$ *and* $(b_1,\ldots,b_l) \in \mathbb{N}^l$ *such that*

$$\langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r = \langle I_0, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r \cap \bigcap_{1\le k\le l} \langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}}\rangle_r,$$

where either $p_k \notin \{j_1,\ldots,j_t\}$, *or* $p_k = j_m$ *and* $b_k < a_m$ *for* $1 \le k \le l$.

Proof. Since $\langle I_0, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r$ is quasi-normal, by Lemma 3.9, it is finitely generated as a radical well-mixed σ-ideal. That is to say, there exist $f_1,\ldots,f_s \in I_0$ such that

$$\langle I_0, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r = \langle f_1,\ldots,f_s, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r.$$

By (4), $I_0 = \cup_{i=0}^{\infty} I^{[i]}$, so there exists $i \in \mathbb{N}$ such that $f_1,\ldots,f_s \in I^{[i]}$. By definition, there exist $g_{i1},\ldots,g_{il_i} \in I^{[i-1]} : T_{j_1\ldots j_t}^{a_1\ldots a_t}$ and $m_{i1},\ldots,m_{il_i} \in T_{j_1\ldots j_t}^{a_1\ldots a_t}$ such that $f_1,\ldots,f_s \in [g_{i1},\ldots,g_{il_i}]$ and $m_{i1}g_{i1},\ldots,m_{il_i}g_{il_i} \in I^{[i-1]}$. There further exist $g_{i-11},\ldots,g_{i-1l_{i-1}} \in I^{[i-2]} : T_{j_1\ldots j_t}^{a_1\ldots a_t}$ and $m_{i-11},\ldots,m_{i-1l_{i-1}} \in T_{j_1\ldots j_t}^{a_1\ldots a_t}$ such that $m_{i1}g_{i1},\ldots,m_{il_i}g_{il_i} \in [g_{i-11},\ldots,g_{i-1l_{i-1}}]$ and $m_{i-11}g_{i-11},\ldots,m_{i-1l_{i-1}}g_{i-1l_{i-1}} \in I^{[i-2]}$. Iterating this process, we eventually have there exist $g_{11},\ldots,g_{1l_1} \in I : T_{j_1\ldots j_t}^{a_1\ldots a_t}$ and $m_{11},\ldots,m_{1l_1} \in T_{j_1\ldots j_t}^{a_1\ldots a_t}$ such that $m_{21}g_{21},\ldots,m_{2l_2}g_{2l_2} \in [g_{11},\ldots,g_{1l_1}]$ and $m_{11}g_{11},\ldots,m_{1l_1}g_{1l_1} \in I$. Hence by Lemma 3.5, we obtain

$$\begin{aligned}\langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r &= \langle I, g_{11},\ldots,g_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r \cap \langle I, m_{11}\cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r\\ &= \langle I, g_{21},\ldots,g_{2l_2}, g_{11},\ldots,g_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r\\ &\quad \cap \langle I, m_{21}\cdots m_{2l_2}m_{11}\cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r\\ &= \cdots\\ &= \langle I, g_{i1},\ldots,g_{il_i},\ldots,g_{11},\ldots,g_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r\\ &\quad \cap \langle I, m_{i1}\cdots m_{il_i}\cdots m_{11}\cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r\\ &= \langle I_0, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r \cap \bigcap_{1\le k\le l} \langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}}\rangle_r\end{aligned}$$

for some $\{p_1,\ldots,p_l\} \subseteq \{1,\ldots,n\}$ and some $(b_1,\ldots,b_l) \in \mathbb{N}^l$, where either $p_k \notin \{j_1,\ldots,j_t\}$, or $p_k = j_m$ and $b_k < a_m$ for $1 \le k \le l$. □

From the proof of Lemma 3.10, we obtain the following lemma which will be used later.

Lemma 3.11. *Suppose that* $\{j_1,\ldots,j_t\} \subseteq \{1,\ldots,n\}$, $(a_1,\ldots,a_t) \in \mathbb{N}^t$ *and* $I \subseteq K\{\mathbb{Y}\}$ *is a pure binomial* σ*-ideal. Let* $h \in N_{j_1\ldots j_t}^{a_1\ldots a_t}(I)\backslash I$. *Then there exist* $\{p_1,\ldots,p_l\} \subseteq \{1,\ldots,n\}$ *and* $(b_1,\ldots,b_l) \in \mathbb{N}^l$ *such that*

$$\langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r = \langle I', y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}\rangle_r \cap \bigcap_{1\le k\le l} \langle I, y_{j_1}^{x^{a_1}}, \ldots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}}\rangle_r,$$

where $I' \supseteq [I, h]$ *is a pure binomial* σ*-ideal and either* $p_k \notin \{j_1,\ldots,j_t\}$, *or* $p_k = j_m$ *and* $b_k < a_m$ *for* $1 \le k \le l$.

Lemma 3.12. *Suppose that* $\{j_1,\dots,j_t\}\subseteq\{1,\dots,n\}$, $(a_1,\dots,a_t)\in\mathbb{N}^t$ *and* $I\subseteq K\{\mathbb{Y}\}$ *is a pure binomial* σ*-ideal. Assume that there exists a binomial* $\mathbb{Y}^{\mathbf{f}}-\mathbb{Y}^{\mathbf{g}}\in I$ *such that* $\mathbb{Y}^{\mathbf{f}}\in[y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$ *and* $\mathbb{Y}^{\mathbf{g}}\notin[y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$. *Then there exist* $\{p_1,\dots,p_l\}\subseteq\{1,\dots,n\}$ *and* $(b_1,\dots,b_l)\in\mathbb{N}^l$ *such that*

$$\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r=\bigcap_{1\le k\le l}\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}},y_{p_k}^{x^{b_k}}\rangle_r,$$

where either $p_k\notin\{j_1,\dots,j_t\}$, *or* $p_k=j_m$ *and* $b_k<a_m$ *for* $1\le k\le l$.

Proof. Since there exists a binomial $\mathbb{Y}^{\mathbf{f}}-\mathbb{Y}^{\mathbf{g}}\in I$ such that $\mathbb{Y}^{\mathbf{f}}\in[y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$ and $\mathbb{Y}^{\mathbf{g}}\notin[y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}]$, then $\mathbb{Y}^{\mathbf{g}}\in\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r$. Therefore, by the properties of radical well-mixed σ-ideals, there exist $\{p_1,\dots,p_l\}\subseteq\{1,\dots,n\}$ and $(b_1,\dots,b_l)\in\mathbb{N}^l$ satisfying either $p_k\notin\{j_1,\dots,j_t\}$, or $p_k=j_m$ and $b_k<a_m$, for $1\le k\le l$ such that $y_{p_1}^{x^{b_1}}\cdots y_{p_l}^{x^{b_l}}\in\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r$. Hence,

$$\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}}\rangle_r=\bigcap_{1\le k\le l}\langle I,y_{j_1}^{x^{a_1}},\dots,y_{j_t}^{x^{a_t}},y_{p_k}^{x^{b_k}}\rangle_r.$$

□

Lemma 3.13. *Let* $i\in\{1,\dots,n\}$ *and* $a\in\mathbb{N}$. *Suppose that* $I\subseteq K\{\mathbb{Y}\}$ *is a pure binomial* σ*-ideal. Then*

$$\langle I,y_i^{x^a}\rangle_r=\bigcap_{(j_1,\dots,j_t),(b_{j_1},\dots,b_{j_t})}\langle I_{j_1\dots j_t}^{b_{j_1}\dots b_{j_t}},y_{j_1}^{x^{b_{j_1}}},\dots,y_{j_t}^{x^{b_{j_t}}}\rangle_r$$

is a finite intersection, where for each member in the intersection, $I_{j_1\dots j_t}^{b_{j_1}\dots b_{j_t}}$ *is a pure binomial* σ*-ideal and either* $I_{j_1\dots j_t}^{b_{j_1}\dots b_{j_t}}\subseteq[y_{j_1}^{x^{b_{j_1}}},\dots,y_{j_t}^{x^{b_{j_t}}}]$, *or* $\langle I_{j_1\dots j_t}^{b_{j_1}\dots b_{j_t}},y_{j_1}^{x^{b_{j_1}}},\dots,y_{j_t}^{x^{b_{j_t}}}\rangle_r$ *is quasi-normal.*

Proof. Use Lemma 3.12 repeatedly and assume that we obtain a decomposition as follows:

$$\langle I,y_i^{x^a}\rangle_r=\bigcap_{(j_1,\dots,j_t),(c_{j_1},\dots,c_{j_t})}\langle I,y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}\rangle_r.\tag{5}$$

For each member in the intersection (5), if $I\subseteq[y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}]$, then we have nothing to do. Otherwise, if there exists a binomial $\mathbb{Y}^{\mathbf{f}}-\mathbb{Y}^{\mathbf{g}}\in I_0\backslash I$ such that $\mathbb{Y}^{\mathbf{f}}\in[y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}]$ and $\mathbb{Y}^{\mathbf{g}}\notin[y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}]$, then by Lemma 3.11,

$$\langle I,y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}\rangle_r=\langle I',y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}\rangle_r\cap\bigcap_{1\le k\le l}\langle I,y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}},y_{p_k}^{x^{d_k}}\rangle_r,$$

where $I'\supseteq[I,\mathbb{Y}^{\mathbf{f}}-\mathbb{Y}^{\mathbf{g}}]$ is a pure binomial σ-ideal and either $p_k\notin\{j_1,\dots,j_t\}$, or $p_k=j_m$ and $d_k<c_{j_m}$ for $1\le k\le l$. Moreover, by Lemma 3.12, we have

$$\langle I',y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}\rangle_r=\bigcap_{1\le k\le l'}\langle I',y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}},y_{s_k}^{x^{e_k}}\rangle_r,$$

where either $s_k\notin\{j_1,\dots,j_t\}$, or $s_k=j_m$ and $e_k<c_{j_m}$ for $1\le k\le l'$. Thus we obtain

$$\langle I,y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}}\rangle_r=\bigcap_{1\le k\le l'}\langle I',y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}},y_{s_k}^{x^{e_k}}\rangle_r\cap\bigcap_{1\le k\le l}\langle I,y_{j_1}^{x^{c_{j_1}}},\dots,y_{j_t}^{x^{c_{j_t}}},y_{p_k}^{x^{d_k}}\rangle_r.\tag{6}$$

By substituting (6) into (5), we rewrite (5) as follows:

$$\langle I, y_i^{x^a} \rangle_r = \bigcap_{(j_1,\ldots,j_t),(c_{j_1},\ldots,c_{j_t})} \langle I_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}} \rangle_r. \tag{7}$$

For each member in the intersection (7), repeat the above process. Let $I_0 = N_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}}(I_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}})$. Since at each step, either the number of elements of $\{y_{j_1}, \ldots, y_{j_t}\}$ strictly increase, or the vector $(c_{j_1}, \ldots, c_{j_t})$ strictly decrease (under the product order), then in finite steps we must obtain either $I_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}} \subseteq [y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}}]$, or for any binomial $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}} \in I_0$, if $\mathbb{Y}^{\mathbf{f}} \in [y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}}]$, then $\mathbb{Y}^{\mathbf{g}} \in [y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}}]$. In the latter case, by Lemma 3.10,

$$\langle I_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}} \rangle_r = \langle I_0, y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}} \rangle_r \cap \bigcap_{1\le k\le l''} \langle I_{j_1\ldots j_t}^{c_{j_1}\cdots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}}, y_{t_k}^{x^{h_k}} \rangle_r,$$

where either $t_k \notin \{j_1, \ldots, j_t\}$, or $t_k = j_m$ and $h_k < c_{j_m}$ for $1 \le k \le l''$. It follows that $\langle I_0, y_{j_1}^{x^{c_{j_1}}}, \ldots, y_{j_t}^{x^{c_{j_t}}} \rangle_r$ is quasi-normal. Apply the same procedure to the rest of the members in the intersection, and in finite steps we eventually obtain the desired decomposition. □

Now we can prove the main theorem of this paper.

Theorem 3.14. *Assume that K is algebraically closed and inversive. Suppose that $I \subseteq K\{\mathbb{Y}\}$ is a pure binomial σ-ideal. Then $\langle I \rangle_r$ is finitely generated as a radical well-mixed σ-ideal.*

Proof. By Lemma 3.7, we have

$$\langle I \rangle_r = \langle I \rangle_r : \mathfrak{m} \cap \langle I, y_{p_1}^{x^{a_1}} \rangle_r \cap \cdots \cap \langle I, y_{p_l}^{x^{a_l}} \rangle_r \tag{8}$$

for some $\{p_1, \ldots, p_l\} \subseteq \{1, \ldots, n\}$ and some $\{a_1, \ldots, a_l\} \in \mathbb{N}^l$. By Lemma 3.13,

$$\langle I, y_{p_k}^{x^{a_k}} \rangle_r = \bigcap_{(j_1,\ldots,j_t),(b_{j_1},\ldots,b_{j_t})} \langle I_{j_1\ldots j_t}^{b_{j_1}\cdots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \ldots, y_{j_t}^{x^{b_{j_t}}} \rangle_r. \tag{9}$$

Since in (9), either $I_{j_1\ldots j_t}^{b_{j_1}\cdots b_{j_t}} \subseteq [y_{j_1}^{x^{b_{j_1}}}, \ldots, y_{j_t}^{x^{b_{j_t}}}]$, or $\langle I_{j_1\ldots j_t}^{b_{j_1}\cdots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \ldots, y_{j_t}^{x^{b_{j_t}}} \rangle_r$ is quasi-normal, then by Lemma 3.9, each member in the intersection (9) is finitely generated as a radical well-mixed σ-ideal. And since (9) is a finite intersection, by Lemma 3.5, $\langle I, y_{p_k}^{x^{a_k}} \rangle_r$ is finitely generated as a radical well-mixed σ-ideal for $1 \le k \le l$. Moreover, by Lemma 3.6, $\langle I \rangle_r : \mathfrak{m}$ is finitely generated as a radical well-mixed σ-ideal. Putting all the above together, by (8) and Lemma 3.5, $\langle I \rangle_r$ is finitely generated as a radical well-mixed σ-ideal. □

Corollary 3.15. *Assume that K is algebraically closed and inversive. Any strictly ascending chain of radical well-mixed σ-ideals generated by pure binomials in $K\{\mathbb{Y}\}$ is finite.*

Proof. Assume that $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \ldots$ is an ascending chain of radical well-mixed σ-ideals generated by pure binomials in $K\{\mathbb{Y}\}$. Then $\cup_{i=1}^{\infty} I_i$ is also a radical well-mixed σ-ideal generated by pure binomials. By Theorem 3.14, $\cup_{i=1}^{\infty} I_i$ is finitely generated as a radical well-mixed σ-ideal, say by $\{a_1, \ldots, a_m\}$. Then there exists $k \in \mathbb{N}$ large enough such that $\{a_1, \ldots, a_m\} \subset I_k$. It follows $I_k = I_{k+1} = \ldots = \cup_{i=1}^{\infty} I_i$. □

Remark 3.16. By Corollary 3.15, Conjecture 1.1 is valid for radical well-mixed σ-ideals generated by pure binomials in a σ-polynomial ring over an algebraic closed and inversive σ-field.

Remark 3.17. Theorem 3.14 and Corollary 3.15 actually hold for radical well-mixed σ-ideals generated by any binomials (not necessarily pure binomials). The proofs are almost identical.

In [6], Levin gave an example to show that a strictly ascending chain of well-mixed σ-ideals in a σ-polynomial ring may be infinite. Here we give a simpler example in terms of well-mixed σ-ideals generated by binomials.

Example 3.18. Let $I = \langle y_1^x y_2 - y_1 y_2^x \rangle$ and $I_0 = [y_1^x y_2 - y_1 y_2^x, y_1^{x^j}(y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l}, y_2^{x^j}(y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l} : i,j,l \in \mathbb{N}, i \geq 2, j \geq i-1]$. We claim that $I = I_0$. It is easy to check that $I_0 \subseteq I$. So we only need to show that I_0 is already a well-mixed σ-ideal. Following Example 3.3, let $\mathcal{I}_L = \langle y_1^x y_2 - y_1 y_2^x \rangle_r$. Suppose $ab \in I_0 \subseteq \mathcal{I}_L$. Since $\mathcal{I}_L = [y_1^{x^i} y_2 - y_1 y_2^{x^i} : i \in \mathbb{N}^*]$ is a σ-prime σ-ideal, then $a \in \mathcal{I}_L$ or $b \in \mathcal{I}_L$. In each case, we can easily deduce $ab^x \in I_0$. Therefore, I_0 is well-mixed and $I = I_0$. So $y_1^{x^2} y_2 - y_1 y_2^{x^2} \notin I$. In fact, in a similar way we can show that $\langle y_1^x y_2 - y_1 y_2^x, \ldots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle = [y_1^x y_2 - y_1 y_2^x, \ldots, y_1^{x^k} y_2 - y_1 y_2^{x^k}, y_1^{x^j}(y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l}, y_2^{x^j}(y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l} : i,j,l \in \mathbb{N}, i \geq k+1, j \geq i-k]$ and $y_1^{x^{k+1}} y_2 - y_1 y_2^{x^{k+1}} \notin \langle y_1^x y_2 - y_1 y_2^x, \ldots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle$ for $k \geq 2$. So we obtain a strictly infinite ascending chain of well-mixed σ-ideals:

$$\langle y_1^x y_2 - y_1 y_2^x \rangle \subsetneq \langle y_1^x y_2 - y_1 y_2^x, y_1^{x^2} y_2 - y_1 y_2^{x^2} \rangle \subsetneq \cdots \subsetneq \langle y_1^x y_2 - y_1 y_2^x, \ldots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle \subsetneq \cdots.$$

As a consequence, $\mathcal{I}_L$ is not finitely generated as a well-mixed σ-ideal.

In [3], it is shown that the radical closure, the reflexive closure, and the perfect closure of a binomial σ-ideal are still a binomial σ-ideal. However, the well-mixed closure of a binomial σ-ideal may not be a binomial σ-ideal. More precisely, it relies on the action of the difference operator. We will give an example to illustrate this.

Example 3.19. Let $K = \mathbb{C}$ and $R = \mathbb{C}\{y_1, y_2, y_3, y_4\}$. Let us consider the σ-ideal $I = \langle y_1^2(y_3 - y_4), y_2^2(y_3 - y_4) \rangle$ of R. Since $(y_1^2 - y_2^2)(y_3 - y_4) = (y_1 + y_2)(y_1 - y_2)(y_3 - y_4) \in I$, we have $(y_1 + y_2)(y_1 - y_2)^x(y_3 - y_4) = (y_1^{x+1} + y_1^x y_2 - y_1 y_2^x - y_2^{x+1})(y_3 - y_4) \in I$. Note that $y_1^{x+1}(y_3 - y_4), y_2^{x+1}(y_3 - y_4) \in I$. Hence $(y_1^x y_2 - y_1 y_2^x)(y_3 - y_4) \in I$. If the difference operator on $\mathbb{C}$ is the identity map, in analogy with Example 4.1 of [7], we can show that $y_1^x y_2(y_3 - y_4), y_1 y_2^x(y_3 - y_4) \notin I$. As a consequence, I is not a binomial σ-ideal.

On the other hand, if the difference operator on $\mathbb{C}$ is the conjugation map (that is $\sigma(i) = -i$), the situation is totally changed. Since $(y_1^2 + y_2^2)(y_3 - y_4) = (y_1 + iy_2)(y_1 - iy_2)(y_3 - y_4) \in I$, $(y_1 + iy_2)(y_1 - iy_2)^x(y_3 - y_4) = (y_1^{x+1} + iy_1^x y_2 + iy_1 y_2^x - y_2^{x+1})(y_3 - y_4) \in I$ and hence $(y_1^x y_2 + y_1 y_2^x)(y_3 - y_4) \in I$. Similarly, we also have $(y_1^x y_2 - y_1 y_2^x)(y_3 - y_4) \in I$. So $y_1^x y_2(y_3 - y_4), y_1 y_2^x(y_3 - y_4) \in I$. Actually $I = [y_1^u(y_3 - y_4)^a, y_1^{w_1} y_2^{w_2}(y_3 - y_4)^a, y_2^v(y_3 - y_4)^a : u, v, w_1, w_2, a \in \mathbb{N}[x], 2 \preceq u, 2 \preceq v, x+1 \preceq w_1 + w_2]$ (the notation $\preceq$ is defined in [7]). In this case, $I = \langle y_1^2(y_3 - y_4), y_2^2(y_3 - y_4) \rangle$ is indeed a binomial σ-ideal.

Remark 3.20. We conjecture that the radical well-mixed closure of a binomial σ-ideal is still a binomial σ-ideal. However, we cannot prove it now.

Acknowledgment

The author thanks C. M. Yuan for helpful discussions.

ORCID

Jie Wang http://orcid.org/0000-0002-9681-1451

References

[1] Eisenbud, D., Sturmfels, B. (1996). Binomial ideals. *Duke Math. J.* 84(1):1–45.
[2] Gao, X. S., Huang, Z., Wang, J., Yuan, C. M. (2017). Toric difference variety. *J. Syst. Sci. Complexity* 2017(30):173–195.
[3] Gao, X. S., Huang, Z., Yuan, C. M. (2017). Binomial difference ideals. *J. Symb. Comput.* 80(3):665–706.
[4] Hrushovski, E. (2012). The elementary theory of the Frobenius automorphisms. Available at: http://www.ma.huji.ac.il/~ehud/, July 2012.
[5] Levin, A. (2008). *Difference Algebra*. New York: Springer-Verlag.
[6] Levin, A. (2015). On the ascending chain condition for mixed difference ideals. *Int. Math. Res. Not.* 2015(10):2830–2840. DOI:10.1093/imrn/rnu021.
[7] Wang, J. (2017). Monomial difference ideals. *Proc. Am. Math. Soc.* 145(4):1481–1496.
[8] Wibmer, M. (2013). *Algebraic Difference Equations*. Lecture Notes. Available in https://www.math.upenn.edu/~wibmer/AlgebraicDifferenceEquations.pdf
[9] Wibmer, M. (2015). Affine Difference algebraic groups. Habilitation thesis, RWTH Aachen University.