Journal of Symbolic Computation 87 (2018) 1–13

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Difference indices of quasi-prime difference algebraic systems

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A R T I C L E I N F O

Article history:
Received 24 July 2016
Accepted 18 May 2017
Available online 26 May 2017

MSC:
primary 12H10
secondary 68W30

Keywords:
Difference index
Order of difference ideal
Hilbert–Levin regularity
Difference ideal membership problem

A B S T R A C T

This paper is devoted to studying difference indices of quasi-prime difference algebraic systems. We define the quasi dimension polynomial of a quasi-prime difference algebraic system. Based on this, we give the definition of the difference index of a quasi-prime difference algebraic system through a family of pseudo-Jacobian matrices. Some properties of difference indices are proved. In particular, an upper bound for difference indices is given. As applications, an upper bound for the Hilbert–Levin regularity and an effective difference ideal membership theorem for quasi-prime difference algebraic systems are deduced.

1. Introduction

The main notion we consider in this paper is the *difference index* of a difference algebraic system over a difference field K (i.e. a field with a transforming operator, for instance $K := \mathbb{Q}(x)$ the field of rational functions with the shift $\sigma : f(x) \mapsto f(x+1)$ as a transforming operator). Roughly speaking, the difference index is an important numerical invariant associated to a difference algebraic system which provides the order of transform we need to apply to the system to obtain the relations up to a prescribed order that all the solutions must verify. In some sense, difference indices can be regarded as a measure of the complexity of difference algebraic systems. The difference index is also closely related to some other important invariants of a difference algebraic system, for example, the order and the Hilbert–Levin regularity. Moreover, difference indices can be utilized to solve the difference ideal membership problem.

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http://dx.doi.org/10.1016/j.jsc.2017.05.004

The analogous notion for a differential algebraic system is the differential index, which has been extensively studied for many years. Actually, there are several definitions of differential indices of a differential algebraic system in the literature (see for instance D'Alfonso et al., 2009, 2008; Campbell and Gear, 1995; Le Vey, 1994; Pantelides, 1988; Seiler, 1999). Although they are not completely equivalent, in each case they represent a measure of the implicitness of the given system. However, the corresponding notion of difference indices for difference algebraic systems has been rarely studied yet. Recently in Wang (2016), difference indices of quasi-regular difference algebraic systems were first defined following the analogous method used in D'Alfonso et al. (2009, 2008) by D'Alfonso, Jeronimo, Massaccesi and Solernó. In this paper, we will generalize the definition of difference indices to more general difference algebraic systems, i.e. quasi-prime difference algebraic systems. The difficulty is to calculate the transcendence degrees of certain associated field extensions without the regular condition. In order to overcome this difficulty, we will introduce the new concept of quasi dimension polynomials for quasi-prime difference algebraic systems. Let us explain it in more details.

Suppose F is a set of difference polynomials, Δ is the difference ideal generated by F, and $\mathfrak{p}$ is a minimal reflexive prime difference ideal over Δ. Denote by Δ_k the algebraic ideal generated by F and the transforms of F up to the order $k-1$ in the corresponding localized polynomial ring at $\mathfrak{p}$. Then we say the system F is *quasi-prime* at $\mathfrak{p}$ if Δ_k is a prime ideal for any positive integer k and Δ is reflexive. For a difference algebraic system F which is quasi-prime at $\mathfrak{p}$, we consider the dimension of Δ_k as a function of k, denoted by $\psi(k)$. It turns out that $\psi(k)$ becomes a polynomial of degree one for k large enough, which we call the $\mathfrak{p}$-*quasi dimension polynomial* of the system F. By virtue of $\mathfrak{p}$-quasi dimension polynomials, we can give the definition of the difference index of a quasi-prime difference algebraic system, which is called the $\mathfrak{p}$-*difference index*. As usual, its definition follows from a certain chain which eventually becomes stationary. In analogy with the case of $\mathfrak{P}$-differential indices in D'Alfonso et al. (2009) and the case of $\mathfrak{p}$-difference indices in Wang (2016), the chain is established by the sequence of ranks of certain Jacobian submatrices associated with the system F. Assume F is quasi-prime at Δ, ω is the Δ-difference index of the system F and ρ is the least k such that the Δ-quasi dimension polynomial of F holds. Then it turns out that for $i+\omega \geq \rho+e-1$ (e is the highest order of F), ω satisfies:

$$\Delta_{i-e+1+\omega} \cap A_i = \Delta \cap A_i,$$

where A_i is the polynomial ring in the variables with orders no more than i, which meets our expectation for difference indices.

This approach enables us to give an upper bound for the $\mathfrak{p}$-difference index of a quasi-prime difference algebraic system. Basing on this, we can give several applications of $\mathfrak{p}$-difference indices, including an upper bound for the Hilbert–Levin regularity and an upper bound of orders for the difference ideal membership problem of a quasi-prime difference algebraic system.

The paper will be organized as follows. In Section 2, we list some basic notions from difference algebra which will be used later. In Section 3, the $\mathfrak{p}$-quasi dimension polynomial of a quasi-prime difference algebraic system is defined. In Section 4, we introduce a family of pseudo-Jacobian matrices and give the definition of $\mathfrak{p}$-difference indices through studying the ranks of them. In Section 5, some properties of $\mathfrak{p}$-difference indices will be proved. In Section 6, several applications of $\mathfrak{p}$-difference indices are given. In Section 7, we give an example.

2. Preliminaries

A *difference ring* or σ-*ring* for short (R, σ), is a commutative ring R together with a ring endomorphism $\sigma: R \to R$. If R is a field, then we call it a *difference field*, or a σ-*field* for short. We call σ the *transforming operator* of R and usually omit σ from the notation, simply refer to R as a σ-ring or a σ-field. A typical example of σ-field is the field of rational functions $\mathbb{Q}(x)$ with $\sigma(f(x)) = f(x+1)$. For any $a \in R$, $\sigma(a)$ is called the *transform* of a. For $n \in \mathbb{N}$, $\sigma^n(a) = \sigma^{n-1}(\sigma(a))$ is called the *n-th transform* of a, with the usual assumption $\sigma^0(a) = a$. In this paper, unless otherwise specified, K is always assumed to be a σ-field of characteristic 0.

Definition 2.1. Let R be a σ-ring. An ideal I of R is called a *σ-ideal* if for $a \in R$, $a \in I$ implies $\sigma(a) \in I$. Suppose I is a σ-ideal of R, then I is said to be

- *reflexive* if $\sigma(a) \in I$ implies $a \in I$ for $a \in R$;
- *σ-prime* if I is reflexive and a prime ideal as an algebraic ideal.

For a subset F in a σ-ring, we denote by $[F]$ the σ-ideal generated by F. Let K be a σ-field. Suppose $\mathbb{Y} = \{y_1, \ldots, y_n\}$ is a set of σ-indeterminates over K. Then the *σ-polynomial ring* over K in $\mathbb{Y}$ is the polynomial ring in the variables $\mathbb{Y}, \sigma(\mathbb{Y}), \sigma^2(\mathbb{Y}), \ldots$. It is denoted by

$$K\{\mathbb{Y}\} = K\{y_1, \ldots, y_n\}$$

and has a natural K-σ-algebra structure. For a σ-polynomial $f \in K\{\mathbb{Y}\}$, the *order* of f, denoted by $\mathrm{ord}(f)$, is the largest j such that the variable $\sigma^j(y_i)$ appears in f for some i. For more details about difference algebra, one can refer to Wibmer (2013).

For the later use, we give the classical Jacobian Criterion here.

Lemma 2.2 *(Jacobian Criterion). Let K be a field of characteristic 0 and $S = K[y_1, \ldots, y_n]$ the polynomial ring over K. Let $I = (f_1, \ldots, f_r)$ be an ideal of S and set $R = S/I$. Let P be a prime ideal of S containing I and assume $\kappa(P)$ is the residue class field of P. Then*

$$\dim_{\kappa(P)} \kappa(P) \otimes \Omega_{R_P/K} = n - \mathrm{rank}_{\kappa(P)} J,$$

where $J := (\partial f_i / \partial y_j)_{r \times n}$ is the Jacobian matrix. In particular, if I is itself a prime ideal, then $\dim_{\kappa(I)} \Omega_{\kappa(I)/K} = n - \mathrm{rank}_{\kappa(I)} J$, where $\kappa(I)$ is the residue class field of I. Here, $\Omega_{R_P/K}$ and $\Omega_{\kappa(I)/K}$ are modules of Kähler differentials.

Proof. One can find a proof in Eisenbud (2004, Chapter 16, Theorem 16.19). □

3. Quasi-prime difference algebraic systems

Let K be a σ-field. Let a be an element in a σ-extension field of K, S a set of elements in a σ-extension field of K, and $i \in \mathbb{N}$. Denote $a^{(i)} = \sigma^i(a), a^{[i]} = \{a, a^{(1)}, \ldots, a^{(i)}\}$, $S^{(i)} = \cup_{a \in S}\{a^{(i)}\}$ and $S^{[i]} = \cup_{a \in S} a^{[i]}$. For the σ-indeterminates $\mathbb{Y} = \{y_1, \ldots, y_n\}$ and $i \in \mathbb{N}$, we will treat the elements of $\mathbb{Y}^{[i]}$ as algebraic indeterminates, and $K[\mathbb{Y}^{[i]}]$ is the polynomial ring in $\mathbb{Y}^{[i]}$ over K.

Throughout the paper let $F = \{f_1, \ldots, f_r\} \subseteq K\{\mathbb{Y}\}$ be a set of difference polynomials over K, $[F]$ the σ-ideal generated by F, and $\mathfrak{p} \subseteq K\{\mathbb{Y}\}$ a σ-prime σ-ideal minimal over $[F]$. Let $e := \max\{\mathrm{ord}(f_i) \mid 1 \le i \le r\}$ for the maximal order of an element of F. We assume that F actually involves the transforming operator, i.e. $e \ge 1$. We also introduce the following auxiliary polynomial rings and ideals: for every $k \in \mathbb{N}$, A_k denotes the polynomial ring $A_k := K[\mathbb{Y}^{[k]}]$ and $\Delta_k := (f_1^{[k-1]}, \ldots, f_r^{[k-1]}) \subseteq A_{k-1+e}$. We set $\Delta_0 := (0)$ by definition.

For each non-negative integer k we write B_k for the local ring obtained from A_k after localization at the prime ideal $A_k \cap \mathfrak{p}$ and let $\mathfrak{p}_k := A_{k-1+e} \cap \mathfrak{p}$. For the sake of simplicity, we preserve the notation Δ_k for the ideal generated by $f_1^{[k-1]}, \ldots, f_r^{[k-1]}$ in the local ring B_{k-1+e} and denote by Δ the σ-ideal generated by F in $K\{\mathbb{Y}\}_{\mathfrak{p}}$.

Definition 3.1. We say that the system F is *quasi-prime* at $\mathfrak{p}$ if Δ_k is a prime ideal in the ring B_{k-1+e} for all $k \in \mathbb{N}$ and Δ is reflexive.

If the system F is quasi-prime at $\mathfrak{p}$, then by the minimality of $\mathfrak{p}$, Δ agrees with $\mathfrak{p}$ in $K\{\mathbb{Y}\}_{\mathfrak{p}}$, since Δ is itself a σ-prime ideal.

Example 3.2. Consider the system $F = \{(y_1^{(1)} + y_2)(y_1^{(1)} - y_2), y_1^{(2)} - y_1\} \subseteq K\{y_1, y_2\}$. Choose $\mathfrak{p} = [y_1^{(1)} - y_2, y_1^{(2)} - y_1]$ which is a σ-prime ideal minimal over $[F]$. For each $k \in \mathbb{N}$, since the generators of

Δ_k is linear in the local ring B_{k-1+e}, Δ_k is clearly prime. To show Δ is reflexive, let $S = K\{y_1, y_2\}\backslash\mathfrak{p}$. Suppose $\sigma(f) \in [F] : S$, then there exists $s \in S$ such that $s\sigma(f) \in [F] \subseteq \mathfrak{p}$. Note that $\mathfrak{p}$ is prime and $s \notin \mathfrak{p}$, so $\sigma(f) \in \mathfrak{p}$ and hence $f \in \mathfrak{p}$. We can write $f = \sum_{i\in I} \alpha_i (y_1^{(1)} - y_2)^{(i)} + \sum_j \beta_{j\in J}(y_1^{(2)} - y_1)^{(j)}$. Let $s' = \prod_{i\in I}(y_1^{(1)} + y_2)^{(i)} \in S$. Then $s'f \in [F]$ and $f \in [F] : S$. So $[F] : S$ is reflexive and by Proposition 1.2.9 of Wibmer (2013), Δ is reflexive. Thus F is quasi-prime at $\mathfrak{p}$.

Remark 3.3. If the σ-ideal $[F] \subseteq K\{\mathbb{Y}\}$ is already a σ-prime σ-ideal, the minimality of $\mathfrak{p}$ implies $\mathfrak{p} = [F]$ and all our results remain true considering the rings A_k and the σ-ideal $[F]$ without localization. In this case if F is quasi-prime at $[F]$, we will say simply that F is quasi-prime.

In this paper, unless otherwise specified, we always assume that F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$.

We will define quasi dimension polynomials for quasi-prime difference algebraic systems. First let us prove a lemma concerning the rank of a certain kind of matrices.

For a matrix E over a σ-field K, we use $E^{(i)}$ to denote the matrix whose elements are the i-th transforms of the corresponding elements of E.

Lemma 3.4. *For a matrix E over a σ-field K, $\mathrm{rank}(E^{(1)}) = \mathrm{rank}(E)$. As a consequence, the dimension of the subspace spanned by the row vectors of $E^{(1)}$ is equal to the dimension of the subspace spanned by the row vectors of E.*

Proof. It is clear that the maximal nonzero minors of $E^{(1)}$ and E have the same order since the transforming operator on K is injective. It follows $\mathrm{rank}(E^{(1)}) = \mathrm{rank}(E)$. $\square$

Lemma 3.5. *Let $E_1, E_2, \ldots, E_t \in K^{p\times q}$ and*

$$M_k := \begin{pmatrix} E_1 & E_2 & \cdots & E_t & & & \\ & E_1^{(1)} & E_2^{(1)} & \cdots & E_t^{(1)} & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & E_1^{(k-1)} & E_2^{(k-1)} & \cdots & E_t^{(k-1)} \end{pmatrix}.$$

Then for k large enough, there exists $d' \in \mathbb{N}$ and $s' \in \mathbb{Z}$ such that

$$\mathrm{rank}(M_k) = d'k + s'. \tag{1}$$

Moreover, the least k such that the equality (1) holds is bounded by $(t-1)(\min\{p, q\} + 1)$.

Proof. For the sake of convenience, for each pair $m, n \in \mathbb{N}, m \le n$, let us define an operator π_n^m on subspaces of K^n,

$$\pi_n^m(V) := \{\mathbf{v} \in K^m \mid (\mathbf{0}, \mathbf{v}) \in V, \mathbf{0} \in K^{n-m}\},$$

where V is a subspace of K^n.

Suppose $k \ge t-1$. Let us apply the row Gaussian elimination to M_k. Step one, apply the Gaussian elimination method to the first $(t-1)p$ rows of M_k, i.e. the submatrix

$$A := \left(\begin{array}{cccc|ccccc|} E_1 & E_2 & \cdots & \cdots & E_t & & & & \\ & E_1^{(1)} & E_2^{(1)} & \cdots & E_{t-1}^{(1)} & E_t^{(1)} & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & E_1^{(t-2)} & E_2^{(t-2)} & \cdots & \cdots & E_t^{(t-2)} & \end{array}\right).$$

Then we obtain a reduced row echelon matrix with several rows containing nonzero elements only in the dotted line area. Denote the submatrix consisting of the fragments of these rows in the dotted line area by B_0. Let U_0 be the subspace of $K^{(t-1)q}$ spanned by the row vectors of B_0.

Step two, apply the Gaussian elimination method to B_0 and the next block submatrix $\begin{pmatrix} E_1^{(t-1)} & E_2^{(t-1)} & \cdots & E_t^{(t-1)} \end{pmatrix}$. We again obtain a reduced row echelon matrix with several rows containing nonzero elements only in the dotted line area:

$$\begin{pmatrix} E_1^{(1)} & E_2^{(1)} & \cdots & \cdots & E_t^{(1)} & & & & \\ & E_1^{(2)} & E_2^{(2)} & \cdots & E_{t-1}^{(2)} & E_t^{(2)} & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & E_1^{(t-1)} & E_2^{(t-1)} & \cdots & \cdots & E_t^{(t-1)} \end{pmatrix}.$$

As before, denote the submatrix consisting of the fragments of these rows in the dotted line area by B_1 and let the row vectors of B_1 span a subspace $U_1 \subseteq K^{(t-1)q}$. If we denote the subspace spanned by the rows of the submatrix A by W, and denote the subspace spanned by the row vectors of $E_1 E_2 \cdots E_t \times 0^{p\times(t-1)q}$ and the vectors in $\{0\}^q \times W^{(1)}$ by P, then $U_0 = \pi_{(2t-2)q}^{(t-1)q}(W)$ and $U_1 = \pi_{(2t-1)q}^{(t-1)q}(P)$. It follows $U_0^{(1)} \subseteq U_1$.

For $j \geq 1$, denote the set of the row vectors of the block submatrix

$$\begin{pmatrix} E_1^{(j+t-2)} & E_2^{(j+t-2)} & \cdots & E_t^{(j+t-2)} \end{pmatrix}$$

by V_j. Then $V_{j+1} = V_j^{(1)}$. Continue performing the Gaussian elimination as step two for each block submatrix $\begin{pmatrix} E_1^{(j+t-2)} & E_2^{(j+t-2)} & \cdots & E_t^{(j+t-2)} \end{pmatrix}$. At each step we obtain a reduced row echelon matrix with several rows containing nonzero elements only in the dotted line area:

$$\begin{pmatrix} E_1^{(j)} & E_2^{(j)} & \cdots & \cdots & E_t^{(j)} & & & & \\ & E_1^{(j+1)} & E_2^{(j+1)} & \cdots & E_{t-1}^{(j+1)} & E_t^{(j+1)} & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & E_1^{(j+t-2)} & E_2^{(j+t-2)} & \cdots & \cdots & E_t^{(j+t-2)} \end{pmatrix}.$$

Denote the submatrix consisting of the fragments of these rows in the dotted line area by B_j and let the row vectors of B_j span a subspace $U_j \subseteq K^{(t-1)q}$. Recursively we have

$$U_j = \pi_{tq}^{(t-1)q}(\mathrm{Span}(U_{j-1} \times \{0\}^q \cup V_j))$$

for $j \geq 1$, where Span represents the spanned linear subspace. Since U_j is the subspace spanned by row vectors of a submatrix with $(t-1)q$ columns and no more than $(t-1)p$ rows, $\dim(U_j) \leq (t-1)\min\{p,q\}$. We next show that the sequence of subspaces $(U_j)_{j\in\mathbb{N}}$ satisfies $U_j^{(1)} \subseteq U_{j+1}$ for all $j \geq 0$ and if $U_j^{(1)} = U_{j+1}$, then $U_{j+1}^{(1)} = U_{j+2}$.

Let us do induction on j. The case $j = 0$ has been proved above. Now suppose $j \geq 1$. Then by the induction hypothesis, $U_j^{(1)} = \pi_{tq}^{(t-1)q}(\mathrm{Span}(U_{j-1}^{(1)} \times \{0\}^q \cup V_j^{(1)})) \subseteq \pi_{tq}^{(t-1)q}(\mathrm{Span}(U_j \times \{0\}^q \cup V_{j+1})) = U_{j+1}$, and if $U_{j-1}^{(1)} = U_j$, then $U_j^{(1)} = U_{j+1}$. So by Lemma 3.4, $\dim(U_j) = \dim(U_j^{(1)}) \leq \dim(U_{j+1})$, and if $\dim(U_j) = \dim(U_{j+1})$, then $\dim(U_{j+1}) = \dim(U_{j+2})$. It follows that $(\dim(U_j))_{j\in\mathbb{N}}$ is a strictly increasing sequence and eventually stabilizes at some constant after at most $(t-1)\min\{p,q\}+1$ steps since the dimensions of the subspaces U_j are no larger than $(t-1)\min\{p,q\}$. So there exists a non-negative integer $r \leq (t-1)\min\{p,q\}$ such that for $j \geq r$, $\dim(U_j) = \dim(U_{j+1})$ and $\dim(\mathrm{Span}(U_j \times \{0\}^q \cup V_{j+1})) = \dim(\mathrm{Span}(U_{j+1} \times \{0\}^q \cup V_{j+2}))$ by Lemma 3.4. Then for $j \geq r$, by the process of the Gaussian elimination, we have

$$\begin{aligned} \mathrm{rank}(M_{j+t}) - \mathrm{rank}(M_{j+t-1}) &= \dim(\mathrm{Span}(U_j \times \{0\}^q \cup V_{j+1})) - \dim(U_j) \\ &= \dim(\mathrm{Span}(U_{j+1} \times \{0\}^q \cup V_{j+2})) - \dim(U_{j+1}) \\ &= \mathrm{rank}(M_{j+t+1}) - \mathrm{rank}(M_{j+t}). \end{aligned}$$

As a consequence, for k large enough, there exist $d', s' \in \mathbb{N}$ such that

$$\mathrm{rank}(M_k) = d'k + s',$$

and the least k such that the above equality holds is bounded by $(t-1)\min\{p,q\} + t - 1 = (t-1)(\min\{p,q\}+1)$. $\square$

Now we introduce quasi dimension polynomials for quasi-prime difference algebraic systems. Define

$$J_k := \frac{\partial(F, F^{(1)}, \ldots, F^{(k-1)})}{\partial(\mathbb{Y}, \mathbb{Y}^{(1)}, \ldots, \mathbb{Y}^{(k-1+e)})} = \begin{pmatrix} \frac{\partial F}{\partial \mathbb{Y}} & \frac{\partial F}{\partial \mathbb{Y}^{(1)}} & \cdots & \frac{\partial F}{\partial \mathbb{Y}^{(e)}} & & & \\ & \frac{\partial F^{(1)}}{\partial \mathbb{Y}^{(1)}} & \frac{\partial F^{(1)}}{\partial \mathbb{Y}^{(2)}} & \cdots & \frac{\partial F^{(1)}}{\partial \mathbb{Y}^{(e+1)}} & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{\partial F^{(k-1)}}{\partial \mathbb{Y}^{(k-1)}} & \frac{\partial F^{(k-1)}}{\partial \mathbb{Y}^{(k)}} & \cdots & \frac{\partial F^{(k-1)}}{\partial \mathbb{Y}^{(k-1+e)}} \end{pmatrix},$$

where each $\frac{\partial F^{(p)}}{\partial \mathbb{Y}^{(q)}}$ denotes the Jacobian matrix $(\partial(f_1^{(p)}, \ldots, f_r^{(p)})/\partial(y_1^{(q)}, \ldots, y_n^{(q)}))_{r\times n}$.

Since the partial derivative operator commutes with the transforming operator, we have

$$J_k = \begin{pmatrix} \frac{\partial F}{\partial \mathbb{Y}} & \frac{\partial F}{\partial \mathbb{Y}^{(1)}} & \cdots & \frac{\partial F}{\partial \mathbb{Y}^{(e)}} & & & \\ & (\frac{\partial F}{\partial \mathbb{Y}})^{(1)} & (\frac{\partial F}{\partial \mathbb{Y}^{(1)}})^{(1)} & \cdots & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(1)} & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & (\frac{\partial F}{\partial \mathbb{Y}})^{(k-1)} & (\frac{\partial F}{\partial \mathbb{Y}^{(1)}})^{(k-1)} & \cdots & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(k-1)} \end{pmatrix}.$$

Denote by $\kappa(\Delta_k)$ the residue class field of Δ_k in the ring B_{k-1+e}, by $\kappa(\mathfrak{p}_k)$ the residue class field of $\mathfrak{p}_k$ in the ring A_{k-1+e}, and by κ the residue class field of $\mathfrak{p}$. To define the $\mathfrak{p}$-quasi dimension polynomial of the system F, we need to add an extra **hypothesis** on the system F: we assume that the rank of the matrix J_k over $\kappa(\Delta_{k+i})$ does not depend on i, where $i \in \mathbb{N}$. That is to say, the rank of the matrix J_k considered alternatively over $\kappa(\Delta_k)$, or over $\kappa(\mathfrak{p}_k)$, or over κ is always the same.

Remark 3.6. This hypothesis is satisfied for relevant classes of difference algebraic systems, for example:

$$F := \begin{cases} f_1 = g_1(\mathbb{Y}, \ldots, \mathbb{Y}^{(e_1)}) & -z_1 \\ & \vdots \\ f_r = g_r(\mathbb{Y}, \ldots, \mathbb{Y}^{(e_r)}) & -z_r \end{cases} \tag{2}$$

where for every $1 \le i \le r$, g_i is a polynomial in the $(e_i+1)n$ variables $\mathbb{Y}, \ldots, \mathbb{Y}^{(e_i)}$ and the variables $\mathbb{Z} = \{z_1, \ldots, z_r\}$ form a new set of σ-indeterminates. Note that $K[\mathbb{Y}^{[k-1+e]}, \mathbb{Z}^{[k-1+e]}]/\Delta_k \simeq K[\mathbb{Y}^{[k-1+e]}]$, so we can regard the entries of J_k as polynomials in the ring $K[\mathbb{Y}^{[k-1+e]}]$. Since $\Delta \cap K[\mathbb{Y}^{[k-1+e]}] = 0$, the ranks of J_k considered over $\kappa(\Delta_k)$ and over κ are the same.

Theorem 3.7. *Suppose F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$. Let $\psi(k) :=$ $\mathrm{trdeg}_K(\kappa(\Delta_k))$. Then for k large enough, there exists $d \in \mathbb{N}$ and $s \in \mathbb{Z}$ such that*

$$\psi(k) = dk + s.$$

Moreover, the least k such that the above equality holds is bounded by $e(\min\{r,n\}+1)$.

Proof. By the property of Kähler differentials, we have $\psi(k) = \mathrm{trdeg}_K(\kappa(\Delta_k)) = \dim_{\kappa(\Delta_k)} \Omega_{\kappa(\Delta_k)/K}$. By Lemma 2.2, $\dim_{\kappa(\mathfrak{p}_k)} \kappa(\mathfrak{p}_k) \otimes \Omega_{\kappa(\Delta_k)/K} = \dim_{\kappa(\Delta_k)} \Omega_{\kappa(\Delta_k)/K} = (k+e)n - \mathrm{rank}_{\kappa(\mathfrak{p}_k)}(J_k) = (k+e)n - \mathrm{rank}_\kappa(J_k)$. It follows $\psi(k) = (k+e)n - \mathrm{rank}_\kappa(J_k)$. Thus the conclusions of the theorem follow from Lemma 3.5 by setting $d = n - d'$ and $s = en - s'$. $\square$

Definition 3.8. In the above theorem, $\psi(k) = dk + s$ is called the $\mathfrak{p}$-*quasi dimension polynomial* of the system F, and the least k such that the $\mathfrak{p}$-quasi dimension polynomial holds is called the $\mathfrak{p}$-*quasi regularity degree* of F, which is denoted by ρ.

4. The definition of $\mathfrak{p}$-difference index

Following D'Alfonso et al. (2009), we introduce a family of pseudo-Jacobian matrices which we need in order to define the concept of $\mathfrak{p}$-difference indices.

Definition 4.1. For each $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$ (i.e. $i \in \mathbb{N}$ and $i \geq e-1$), we define the $kr \times kn$-matrix $J_{k,i}$ as follows:

$$J_{k,i} := \frac{\partial(F^{(i-e+1)}, F^{(i-e+2)}, \ldots, F^{(i-e+k)})}{\partial(\mathbb{Y}^{(i+1)}, \mathbb{Y}^{(i+2)}, \ldots, \mathbb{Y}^{(i+k)})} = \begin{pmatrix} \frac{\partial F^{(i-e+1)}}{\partial \mathbb{Y}^{(i+1)}} & 0 & \cdots & 0 \\ \frac{\partial F^{(i-e+2)}}{\partial \mathbb{Y}^{(i+1)}} & \frac{\partial F^{(i-e+2)}}{\partial \mathbb{Y}^{(i+2)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^{(i-e+k)}}{\partial \mathbb{Y}^{(i+1)}} & \frac{\partial F^{(i-e+k)}}{\partial \mathbb{Y}^{(i+2)}} & \cdots & \frac{\partial F^{(i-e+k)}}{\partial \mathbb{Y}^{(i+k)}} \end{pmatrix},$$

where each $\frac{\partial F^{(p)}}{\partial \mathbb{Y}^{(q)}}$ denotes the Jacobian matrix $(\partial(f_1^{(p)}, \ldots, f_r^{(p)})/\partial(y_1^{(q)}, \ldots, y_n^{(q)}))_{r\times n}$.

Since the partial derivative operator commutes with the transforming operator, we have

$$J_{k,i} = \begin{pmatrix} (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(i-e+1)} & 0 & \cdots & 0 \\ (\frac{\partial F}{\partial \mathbb{Y}^{(e-1)}})^{(i-e+2)} & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(i-e+2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{\partial F}{\partial \mathbb{Y}^{(e-k+1)}})^{(i-e+k)} & (\frac{\partial F}{\partial \mathbb{Y}^{(e-k+2)}})^{(i-e+k)} & \cdots & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(i-e+k)} \end{pmatrix},$$

where we set that $\frac{\partial F}{\partial \mathbb{Y}^{(j)}} = 0$ if $j < 0$.

Note that $J_{k,i+1} = J_{k,i}^{(1)}$.

Definition 4.2. For $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$, we define $\mu_{k,i} \in \mathbb{N}$ as follows:

- $\mu_{0,i} := 0$;
- $\mu_{k,i} := \dim_\kappa \ker(J_{k,i}^\tau)$, for $k \geq 1$, where $J_{k,i}^\tau$ denotes the usual transpose of the matrix $J_{k,i}$. In particular $\mu_{k,i} = kr - \mathrm{rank}_\kappa(J_{k,i})$.

Proposition 4.3. *Let $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$. Then $\mu_{k,i} = \mu_{k,i+1}$.*

Proof. Since $J_{k,i+1} = J_{k,i}^{(1)}$ for any $k \in \mathbb{N}$ and any $i \in \mathbb{N}_{\geq e-1}$, then $\mu_{k,i} = \mu_{k,i+1}$ follows from Lemma 3.4. □

The previous proposition shows that the sequence $\mu_{k,i}$ does not depend on the index i. Therefore, in the sequel, we will write μ_k instead of $\mu_{k,i}$, for any $i \in \mathbb{N}_{\geq e-1}$.

For $k \in \mathbb{N}$ and $i \in \mathbb{N}_{\geq e-1}$, we denote by $\Omega_{i,k}$ the residue class field of $\Delta_{i-e+1+k} \cap B_i$ in the ring B_i. As an additional **hypothesis** on the system F, we assume that the rank of the matrix $J_{k,i}$ over $\kappa(\Delta_{i-e+1+k+s})$ does not depend on s, where $s \in \mathbb{N}$. That is to say, we assume that the rank of the matrix $J_{k,i}$ considered alternatively over $\kappa(\Delta_{i-e+1+k})$, or over $\kappa(\mathfrak{p}_{i-e+1+k})$, or over κ is always the same.

Remark 4.4. This hypothesis is satisfied for relevant classes of difference algebraic systems such as (2) in Remark 3.6. Note that $K[\mathbb{Y}^{[i+k]}, \mathbb{Z}^{[i-e+k]}]/\Delta_{i-e+1+k} \simeq K[\mathbb{Y}^{[i+k]}]$, so we can regard the entries of $J_{k,i}$ as polynomials in the ring $K[\mathbb{Y}^{[i+k]}]$. Since $\Delta \cap K[\mathbb{Y}^{[i+k]}] = 0$, the ranks of $J_{k,i}$ considered over $\kappa(\Delta_{i-e+1+k})$ and over κ are the same. The analogous hypothesis is also required in various notions of differentiation indices.

Proposition 4.5. *Assume that the* $\mathfrak{p}$*-quasi dimension polynomial of* F *is* $\psi(k) = dk + s$ *and the* $\mathfrak{p}$*-quasi regularity degree is* ρ*. Let* $k \in \mathbb{N}$ *and* $i \in \mathbb{N}_{\geq e-1}$*. Then*

1. *The transcendence degree of the field extension*

$$\mathrm{Frac}(B_i/(\Delta_{i-e+1+k} \cap B_i)) \hookrightarrow \mathrm{Frac}(B_{i+k}/\Delta_{i-e+1+k})$$

is $k(n-r) + \mu_k$*.*

2. *For* $i + k \geq \rho + e - 1$*, the following identity holds:*

$$\mathrm{trdeg}_K(\mathrm{Frac}(B_i/(\Delta_{i-e+1+k}) \cap B_i)) = d(i+1) + (d+r-n)k + s - ed - \mu_k.$$

Proof. 1. We can consider the field $\mathrm{Frac}(B_{i+k}/\Delta_{i-e+1+k})$ as the fraction field of

$$\Omega_{i,k}[\mathbb{Y}^{(i+1)}, \ldots, \mathbb{Y}^{(i+k)}]/(F^{(i-e+1)}, \ldots, F^{(i-e+k)}).$$

Therefore by the property of Kähler differentials and Lemma 2.2, the transcendence degree of the field extension equals $kn - \mathrm{rank}_\kappa(J_{k,i}) = kn - (kr - \mu_k) = k(n-r) + \mu_k$.

2. Since when $i+k \geq \rho + e - 1$, by Theorem 3.7, $\mathrm{trdeg}_K(\mathrm{Frac}(B_{i+k}/\Delta_{i-e+1+k})) = d(i-e+1+k)+s$, we have

$$\begin{aligned}\mathrm{trdeg}_K(\mathrm{Frac}(B_i/(\Delta_{i-e+1+k} \cap B_i))) &= d(i-e+1+k) + s - k(n-r) - \mu_k \\ &= d(i+1) + (d+r-n)k + s - ed - \mu_k. \quad \square\end{aligned}$$

We prove another lemma concerning the rank of a certain kind of matrices.

Lemma 4.6. *Let* $E_1, E_2, \ldots, E_t \in K^{p\times q}$ *and*

$$N_k := \begin{pmatrix} E_1 & & & & \\ E_2^{(1)} & E_1^{(1)} & & & \\ \vdots & \vdots & \ddots & & \\ E_t^{(t-1)} & E_{t-1}^{(t-1)} & \cdots & E_1^{(t-1)} & \\ & \ddots & \ddots & \ddots & \ddots \\ & & E_t^{(k-1)} & E_{t-1}^{(k-1)} & \cdots & E_1^{(k-1)} \end{pmatrix}.$$

Then for k *large enough, there exists* $d' \in \mathbb{N}$ *and* $a' \in \mathbb{Z}$ *such that*

$$\mathrm{rank}(N_k) = d'k + a'.$$

Moreover, the least k *such that the above equality holds is bounded by* $(t-1)(\min\{p,q\}+2)$*.*

Proof. Assume $k \geq 2t-2$. Denote the submatrix consisting of the first $(t-1)p$ rows and the first $(t-1)q$ columns of N_k by A, that is

$$A := \begin{pmatrix} E_1 & & & \\ E_2^{(1)} & E_1^{(1)} & & \\ \vdots & \vdots & \ddots & \\ E_{t-1}^{(t-2)} & E_{t-2}^{(t-2)} & \cdots & E_1^{(t-2)} \end{pmatrix},$$

and denote the submatrix of N_k by removing the first $(t-1)p$ rows by C_k, that is

$$C_k := \begin{pmatrix} E_t^{(t-1)} & E_{t-1}^{(t-1)} & \cdots & E_1^{(t-1)} & & & \\ & E_t^{(t)} & E_{t-1}^{(t)} & \cdots & E_1^{(t)} & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & E_t^{(k-1)} & E_{t-1}^{(k-1)} & \cdots & E_1^{(k-1)} \end{pmatrix}.$$

In analogy with the proof of Lemma 3.5, apply the Gaussian elimination method to C_k, but from bottom to top and from right to left. Then, for k large enough, there exists $d' \in \mathbb{N}$ and $s' \in \mathbb{Z}$ such that

$$\operatorname{rank}(C_k) = d'(k-t+1) + s',$$

and the least k such that $\operatorname{rank}(C_k) = d'(k-t+1) + s'$ is bounded by $(t-1)(\min\{p,q\}+1) + t - 1 = (t-1)(\min\{p,q\}+2)$. And we obtain a reduced row echelon matrix with several rows containing nonzero elements only in the first $(t-1)q$ columns. Denote the submatrix consisting of the fragments of these rows in the first $(t-1)q$ columns by B. One can see that B does not rely on k for k large enough. Perform the Gaussian elimination method to the submatrix A by using the row vectors of B and it follows that for k large enough, there exists a constant $c \in \mathbb{N}$ such that $\operatorname{rank}(N_k) = \operatorname{rank}(C_k) + c$. Hence, $\operatorname{rank}(N_k) = d'(k-t+1) + s' + c$ for k large enough. Set $a' = -d'(t-1) + s' + c$. So for k large enough, $\operatorname{rank}(N_k) = d'k + a'$ and the least k such that $\operatorname{rank}(N_k) = d'k + a'$ is bounded by $(t-1)(\min\{p,q\}+2)$. $\square$

Due to Lemma 4.6, we can prove a formula of μ_k for $k \gg 0$. (We use $k \gg 0$ to denote k large enough.)

Theorem 4.7. *Suppose F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$. Assume the $\mathfrak{p}$-quasi dimension polynomial of F is $\psi(k) = dk + s$. Then for $k \gg 0$, there exists $a \in \mathbb{Z}$ such that*

$$\mu_k = (d + r - n)k + a. \tag{3}$$

Moreover, an upper bound of the least k such that the equality (3) holds is $e(\min\{r,n\}+2)$.

Proof. Set $i = e - 1$. Then for $k \gg 0$,

$$J_{k,e-1} = \begin{pmatrix} \frac{\partial F}{\partial \mathbb{Y}^{(e)}} & & & & \\ (\frac{\partial F}{\partial \mathbb{Y}^{(e-1)}})^{(1)} & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(1)} & & & \\ \vdots & \vdots & \ddots & & \\ (\frac{\partial F}{\partial \mathbb{Y}})^{(e)} & (\frac{\partial F}{\partial \mathbb{Y}^{(1)}})^{(e)} & \cdots & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(e)} & \\ & \ddots & \ddots & \ddots & \ddots \\ & & (\frac{\partial F}{\partial \mathbb{Y}})^{(k-1)} & (\frac{\partial F}{\partial \mathbb{Y}^{(1)}})^{(k-1)} & \cdots & (\frac{\partial F}{\partial \mathbb{Y}^{(e)}})^{(k-1)} \end{pmatrix}.$$

So by Lemma 4.6, for $k \gg 0$, there exists $d' \in \mathbb{N}$ and $a' \in \mathbb{Z}$ such that $\operatorname{rank}(J_{k,e-1}) = d'k + a'$, and the least k such that $\operatorname{rank}(J_{k,e-1}) = d'k + a'$ is bounded by $e(\min\{r,n\}+2)$. Note that $d' = n - d$. Set $a = -a'$. Hence for $k \gg 0$, $\mu_k = kr - \operatorname{rank}(J_{k,e-1}) = (d + r - n)k + a$, and an upper bound of the least k such that $\mu_k = (d + r - n)k + a$ is $e(\min\{r,n\}+2)$. $\square$

Remark 4.8. Let ρ be the $\mathfrak{p}$-quasi regularity degree of the system F. From the proof of Lemma 4.6, we actually have a more accurate upper bound for the least k such that $\mu_k = (d + r - n)k + a$, namely, $\rho + e$.

Remark 4.9. In fact, we can deduce the formula of μ_k for $k \gg 0$ in a more straightforward way. Fix an index $i \in \mathbb{N}_{\geq e-1}$. By Proposition 4.5, for $k \gg 0$, we have $\psi(i-e+1+k) = k(n-r) + \mu_k + \mathrm{trdeg}_K(\Omega_{i,k})$. Note that $\mathrm{trdeg}_K(\Omega_{i,k})$ will be a constant for $k \gg 0$ since the increasing chain $(\Delta_{i-e+1+k} \cap B_i)_{k\in\mathbb{N}}$ of prime ideals in the ring B_i is stable. So by Theorem 3.7, μ_k is a polynomial of degree one for $k \gg 0$.

Definition 4.10. In Theorem 4.7, the least integer k such that $\mu_k = (d+r-n)k + a$ is called the $\mathfrak{p}$-*difference index* of the system F, which is denoted by ω. If $[F]$ is itself a σ-prime σ-ideal, we say simply the difference index of F.

It is obvious from the construction that ω is depending on the choice of the minimal σ-prime σ-ideal $\mathfrak{p}$ over $[F]$. However, we will prove some properties of ω which meet our expectation for difference indices.

5. Properties of $\mathfrak{p}$-difference index

A notable property of most differentiation indices is that they provide an upper bound for the number of derivatives of the system needed to obtain all the equations that must be satisfied by the solutions of the system. This case is also suitable for the $\mathfrak{p}$-difference indices defined above.

Theorem 5.1. *Suppose F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$. Let ρ and ω be the $\mathfrak{p}$-quasi regularity degree and the $\mathfrak{p}$-difference index of the system F respectively. Then, for $i \in \mathbb{N}_{\geq e-1}$ such that $i+\omega \geq \rho+e-1$, the equality of ideals*

$$\Delta_{i-e+1+\omega} \cap B_i = \Delta \cap B_i$$

holds in the ring B_i. Moreover, for every $i \in \mathbb{N}_{\geq e-1}$, let $h_i := \min\{h \in \mathbb{N} : \Delta_{i-e+1+h} \cap B_i = \Delta \cap B_i\}$. If $i+\omega \geq \rho+e-1$ and $i+h_i \geq \rho+e-1$, then $\omega = h_i$.

Proof. The proof is similar to Theorem 5.1 of Wang (2016) and we omit it. □

Remark 5.2. Taking $i = e-1$ in the last assertion of the above theorem, we obtain that if $\omega \geq \rho$ and $h_{e-1} \geq \rho$, then one has the following equality for the $\mathfrak{p}$-difference index:

$$\omega = \min\{h \in \mathbb{N} : \Delta_h \cap B_{e-1} = \Delta \cap B_{e-1}\}.$$

The following proposition reveals a connection between the formula of μ_k for $k \gg 0$ and the properties of the dimension polynomial of $\mathfrak{p}$ (see Section 6.1).

Proposition 5.3. *Suppose F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$. Assume the $\mathfrak{p}$-quasi dimension polynomial of F is $\psi(k) = dk + s$ and for $k \gg 0$, $\mu_k = (d+r-n)k + a$. Then $d = \sigma\text{-}\dim(\mathfrak{p})$ and $a = s - ed - \mathrm{ord}(\mathfrak{p})$, where $\sigma\text{-}\dim(\mathfrak{p})$ and $\mathrm{ord}(\mathfrak{p})$ are the difference dimension and the order of $\mathfrak{p}$ respectively. In particular, if ω is the $\mathfrak{p}$-difference index of the system F, then $\mu_\omega = (d+r-n)\omega + s - ed - \mathrm{ord}(\mathfrak{p})$.*

Proof. Let ρ be the $\mathfrak{p}$-quasi regularity degree of the system F. Fix an index $i \in \mathbb{N}_{\geq e-1}$ such that $i+\omega \geq \rho+e-1$. By Theorem 5.1, for $k \geq \omega$, $\Delta_{i-e+1+k} \cap B_i = \Delta \cap B_i$. Therefore, for $k \geq \omega$, by Proposition 4.5 and Theorem 4.7,

$$\begin{aligned}\mathrm{trdeg}_K(\mathrm{Frac}(B_i/(\Delta \cap B_i))) &= \mathrm{trdeg}_K(\mathrm{Frac}(B_i/(\Delta_{i-e+1+k} \cap B_i))) \\ &= d(i+1) + (d+r-n)k + s - ed - \mu_k \\ &= d(i+1) + s - ed - a.\end{aligned}$$

On the other hand, since $\mathrm{Frac}(B_i/(\Delta \cap B_i)) = \mathrm{Frac}(A_i/(\mathfrak{p} \cap A_i))$, by Wibmer (2013), Section 5.1,

$$\mathrm{trdeg}_K(\mathrm{Frac}(B_i/(\Delta \cap B_i))) = \sigma\text{-}\dim(\mathfrak{p})(i+1) + \mathrm{ord}(\mathfrak{p}).$$

So

$$d(i+1)+s-ed-a=\sigma\text{-}\dim(\mathfrak{p})(i+1)+\mathrm{ord}(\mathfrak{p}), \tag{4}$$

for all $i \in \mathbb{N}_{e-1}$ such that $i+\omega \geq \rho+e-1$. Compare the coefficients of i on the two sides of the identity (4), and it follows $d=\sigma\text{-}\dim(\mathfrak{p})$ and $a=s-ed-\mathrm{ord}(\mathfrak{p})$. □

Remark 5.4. Note that $\Delta_{i-e+1} \subseteq \Delta \cap B_i$, so for $i \geq \rho+e-1$, we have $\psi(i-e+1)=d(i-e+1)+s \geq d(i+1)+\mathrm{ord}(\mathfrak{p})$ and hence $s \geq ed+\mathrm{ord}(\mathfrak{p})$. Therefore, by Proposition 5.3, $a=s-ed-\mathrm{ord}(\mathfrak{p}) \geq 0$.

6. Applications of $\mathfrak{p}$-difference index

6.1. The Hilbert–Levin regularity

For a σ-prime σ-ideal $\mathfrak{p}$, the polynomial $\varphi(i)=\sigma\text{-}\dim(\mathfrak{p})(i+1)+\mathrm{ord}(\mathfrak{p})$ is known as the dimension polynomial of $\mathfrak{p}$ (see for instance Wibmer, 2013, Chapter 5). The minimum of the indices i_0 such that $\varphi(i)=\mathrm{trdeg}_K(\mathrm{Frac}(A_i/(A_i \cap \mathfrak{p})))$ for all $i \geq i_0$ is called the *Hilbert–Levin regularity* of $\mathfrak{p}$. The results developed on $\mathfrak{p}$-difference indices enable us to give an upper bound for the Hilbert–Levin regularity of $\mathfrak{p}$.

Theorem 6.1. *Suppose F is a difference algebraic system which is quasi-prime at $\mathfrak{p}$. Let ρ and ω be the $\mathfrak{p}$-quasi regularity degree and the $\mathfrak{p}$-difference index of the system F respectively. Then the Hilbert–Levin regularity of the σ-prime σ-ideal $\mathfrak{p}$ is bounded by $e-1+\max\{0, \rho-\omega\}$.*

Proof. The proof is similar to Theorem 6.1 of Wang (2016) with a little change and we omit it. □

6.2. The ideal membership problem

It is well known that in polynomial algebra, the ideal membership problem is to decide if a given element $f \in A$ belongs to a fixed ideal $I \subseteq A$ for a polynomial ring A, and if the answer is yes, to represent f as a linear combination with polynomial coefficients of a given set of generators of I.

The ideal membership problem also exists in differential algebra and difference algebra. But unlike the case in polynomial algebra, this problem is undecidable for arbitrary ideals in differential algebra (see Gallo et al., 1991) and difference algebra. However, there are special classes of differential ideals for which the problem is decidable, in particular the class of radical differential ideals (Seidenberg, 1956, see also Boulier et al., 1995). By virtue of Theorem 5.1, we are able to give an order bound for the ideal membership problem of a quasi-prime difference algebraic system.

The following ideal membership theorem for polynomial rings will be used.

Theorem 6.2. *(Aschenbrenner, 2004, Theorem 3.4) Let K be a field and $g, g_1, \ldots, g_s \in K[y_1, \ldots, y_n]$ be a set of polynomials whose total degrees are bounded by an integer d. If g is a polynomial belonging to the ideal generated by $g_1, \ldots, g_s$, then there exist polynomials $a_1, \ldots, a_s$ such that $g=\sum_{j=1}^{s} a_j g_j$ and $\deg(a_j) \leq (2d)^{2^n}$ for $1 \leq j \leq s$.*

Now we obtain the following effective ideal membership theorem for quasi-prime difference algebraic systems:

Theorem 6.3. *Suppose F is a quasi-prime difference algebraic system in the sense of Remark 3.3. Let ρ and ω be the quasi regularity degree and the difference index of the system F respectively. Let $f \in K\{\mathbb{Y}\}$ be any σ-polynomial in the σ-ideal $[F]$ such that $\omega+\max\{0, \mathrm{ord}(f)-e+1\} \geq \rho$. Let D be an upper bound for the total degrees of $f, f_1, \ldots, f_r$. Set $N := \omega+\max\{-1, \mathrm{ord}(f)-e\}$. Then, a representation*

$$f=\sum_{1 \leq i \leq r, 0 \leq j \leq N} g_{ij} f_i^{(j)}$$

holds in the ring A_{N+e}, where polynomials g_{ij} have total degrees bounded by $(2D)^{2^{(N+e+1)n}}$.

Proof. The upper bound on the order of transforms needed to apply to $f_1, \dots, f_r$ is a direct consequence of Theorem 5.1 applied to $i := \max\{e-1, \mathrm{ord}(f)\}$. The degree upper bound for the polynomials g_{ij} follows from Theorem 6.2. $\square$

Remark 6.4. Since we have an upper bound $e(\min\{r,n\}+2)$ for ω, it suffices to take $N = e(\min\{r,n\}+2) + \max\{-1, \mathrm{ord}(f) - e\}$ to get more explicit upper bounds of the order and the degree in the above ideal membership problem.

7. An example

Example 7.1. Notations follow as before. Consider the difference algebraic system $F = \{y_1^{(2)} - y_1, y_1^{(1)} - y_2, y_1y_2 - 1\} \subseteq A = K\{y_1, y_2\}$. Then $\Delta = [F]$ is a σ-prime σ-ideal and F is a quasi-prime system in the sense of Remark 3.3. We have $n = 2, r = 3, e = 2, d = 0$. The corresponding matrices $J_k, k = 1, 2, 3, \dots$ are

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 1 & 0 & & & & & \\
0 & -1 & 1 & 0 & 0 & 0 & & & & & \\
y_2 & y_1 & 0 & 0 & 0 & 0 & & & & & \\
 & & -1 & 0 & 0 & 0 & 1 & 0 & & & \\
 & & 0 & -1 & 1 & 0 & 0 & 0 & & & \\
 & & y_1 & y_2 & 0 & 0 & 0 & 0 & & & \\
 & & & & -1 & 0 & 0 & 0 & 1 & 0 & \\
 & & & & 0 & -1 & 1 & 0 & 0 & 0 & \\
 & & & & y_2 & y_1 & 0 & 0 & 0 & 0 & \\
 & & & & & & \cdots & & \cdots & & \cdots
\end{pmatrix},$$

and $J_{k,1}, k = 1, 2, 3, \dots$ are

$$\begin{pmatrix}
1 & 0 & & & & & & & & & \\
0 & 0 & & & & & & & & & \\
0 & 0 & & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & \\
1 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & \\
-1 & 0 & 0 & 0 & 1 & 0 & & & & & \\
0 & -1 & 1 & 0 & 0 & 0 & & & & & \\
y_2 & y_1 & 0 & 0 & 0 & 0 & & & & & \\
 & & -1 & 0 & 0 & 0 & 1 & 0 & & & \\
 & & 0 & -1 & 1 & 0 & 0 & 0 & & & \\
 & & y_1 & y_2 & 0 & 0 & 0 & 0 & & & \\
 & & & & -1 & 0 & 0 & 0 & 1 & 0 & \\
 & & & & 0 & -1 & 1 & 0 & 0 & 0 & \\
 & & & & y_2 & y_1 & 0 & 0 & 0 & 0 & \\
 & & & & & & \cdots & & \cdots & & \cdots
\end{pmatrix}.$$

Since $y_1^{(2i)} = y_1, y_1^{(2i+1)} = y_2, y_2^{(2i)} = y_2, y_2^{(2i+1)} = y_1$ in the ring A/Δ for all $i \in \mathbb{N}$, we have replaced $y_1^{(2i)}, y_1^{(2i+1)}, y_2^{(2i)}, y_2^{(2i+1)}$ by y_1, y_2, y_2, y_1 respectively in J_k and $J_{k,1}$ for all $i \in \mathbb{N}$. It can be computed that $\mathrm{rank}(J_1) = 3$, $\mathrm{rank}(J_2) = 5$, $\mathrm{rank}(J_3) = 7$. In fact, $\mathrm{rank}(J_k) = 2k + 1$ for all $k \geq 1$. So the quasi dimension polynomial of the system F is $\psi(k) = 2k + 1$ and the quasi regularity degree $\rho = 1$. Also, one can compute that $\mathrm{rank}(J_{1,1}) = 1$, $\mathrm{rank}(J_{2,1}) = 2$, $\mathrm{rank}(J_{3,1}) = 4$, $\mathrm{rank}(J_{4,1}) = 6$, so $\mu_1 = 2, \mu_2 = 4, \mu_3 = 5, \mu_4 = 6$. In fact, $\mu_k = k + 2$ for all $k \geq 2$. Hence the difference index of the system F is $\omega = 2$. One can check that $\Delta_2 \cap A_1 = \Delta \cap A_1$.

Acknowledgements

The author thanks Li Wei for helpful suggestions.

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