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Nonnegative Polynomials and Circuit Polynomials*

Jie Wang†

Abstract. The concept of sums of nonnegative circuit (SONC) polynomials was recently introduced as a new certificate of nonnegativity especially for sparse polynomials. In this paper, we explore the relationship between nonnegative polynomials and SONCs. As a first result, we provide sufficient conditions for nonnegative polynomials with general Newton polytopes to be a SONC, which generalizes the previous result on nonnegative polynomials with simplex Newton polytopes. Second, we prove that every SONC admits a SONC decomposition without cancellation. In other words, SONC decompositions preserve sparsity of nonnegative polynomials, which is dramatically different from the classical sum of squares decompositions and is a key property to design efficient algorithms for sparse polynomial optimization based on SONC decompositions.

Key words. nonnegative polynomial, sum of nonnegative circuit polynomials, SONC, certificate of nonnegativity, sum of squares, SAGE

AMS subject classifications. 14P10, 90C23, 90C26, 12D10, 12D15

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1. Introduction.

A real polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called a *nonnegative polynomial* if its evaluation on every real point is nonnegative. All nonnegative polynomials form a convex cone, denoted by PSD. Certifying nonnegativity of multivariate polynomials is a central problem of real algebraic geometry and also has a deep connection with polynomial optimization. A classical approach for handling this problem is using sum of squares (SOS) decompositions. From the perspective of computation, a common algorithm for checking whether a polynomial admits an SOS decomposition relies on a semidefinite program (SDP) involving a positive semidefinite matrix of size $\binom{n+d}{n}$, where n is the number of variables and $2d$ is the degree of the polynomial [15]. Hence, the size of the corresponding SDP problem increases combinatorially with n, d, which greatly limits the scalability of this approach given the current state of SDP solvers. To address the issue of scalability, one possibility is to exploit the structure in the polynomial data, such as symmetry [3], correlative sparsity [21], term sparsity [23, 25, 26], and correlative-term sparsity [27], just to name a few. Another possibility is to rely on other nonnegativity certificates. Such alternative nonnegativity certificates are in general more restrictive but cheaper to implement, e.g., (scaled) diagonally dominant sums of squares [1]. However, a common drawback shared by these approaches is that their computational complexity depends on the polynomial degree. As an attempt to overcome this, Iliman and de Wolff proposed the concept of *sums of nonnegative circuit*

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†Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China (wangjie212@amss.ac.cn, https://wangjie212.github.io/jiewang/).

(SONC) polynomials as a new nonnegativity certificate of polynomials [5]. A *circuit polynomial* is of the form $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, where $c_{\boldsymbol{\alpha}} > 0$ for all $\boldsymbol{\alpha} \in \mathscr{A}$, $\mathscr{A} \subseteq (2\mathbb{N})^n$ comprises the vertices of a simplex, and $\boldsymbol{\beta}$ lies in the relative interior of this simplex. The support of a circuit polynomial is called a *circuit*. The study of circuit polynomials dates back to 1980s by Reznick [18] in the special case of simplicial agiforms. After over two decades of quiescence, a nonnegativity condition for circuit polynomials was given by Paneta, Koeppl, and Craciun in the study of biochemical reaction networks [13], and the subject was brought back to people's view. A related certificate, *sums of AGE (SAGE) polynomials*, was also recently proposed by Murray, Chandrasekaranin and Wiermann, [9], where an AGE polynomial is defined by a nonnegative polynomial with at most one term that can take negative values (called a negative term). The set of nonnegative polynomials that admit SONC decompositions forms a convex cone, i.e., the SONC cone, and the set of nonnegative polynomials that admit SAGE decompositions forms a convex cone, i.e., the SAGE cone. SONC has been used to solve sparse polynomial optimization via geometric programming [2, 6, 14, 20] by Dressler and others or via second order cone programming [8, 24] by the author and Magron. SAGE has been used to solve sparse polynomial/signomial optimization via relative entropy programming [9, 10] by Murray, Chandrasekaran, and Wiermann. From the perspective of theory, it is then natural to ask the following:

1. Which types of nonnegative polynomials lie in the SONC cone? Can we provide sufficient conditions for a nonnegative polynomial to admit a SONC decomposition in terms of the support?
2. What is the relationship bewtween the SONC cone and the SAGE cone?

In [5], Iliman and de Wolff proved that if the Newton polytope of a polynomial f is a simplex and there exists a point such that all terms of f except for those corresponding to the vertices of the Newton polytope take negative values on this point, then f is nonnegative if and only if f admits a SONC decomposition (see Theorem 2.8). The first contribution of the present paper is that we generalize this conclusion to polynomials with general Newton polytopes. In particular, we prove that a polynomial with one negative term is nonnegative if and only if it admits a SONC decomposition (Theorem 1.1).

Theorem 1.1. *Let $f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d_0\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\dots,m$. Then f is nonnegative if and only if f lies in the SONC cone.*

Note that Theorem 1.1 tells us that any AGE polynomial admits a SONC decomposition. As an immediate corollary, we obtain that the SAGE cone and the SONC cone are actually identical. Taking a step further, we also provide sufficient conditions for nonnegative polynomials with multiple negative terms admitting a SONC decomposition in terms of the combinatorial structure of supports (Theorem 1.2). Below we say that a vertex of a polytope is simple if this vertex lies on precisely d edges with d being the dimension of the polytope.

Theorem 1.2. *Let $f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\dots,m$, and $d_j < 0, j = 1,\dots,l$. Assume that some vertex of the Newton polytope of f is simple and all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ (in the affine subspace spanned by the support of f). Then f is nonnegative if and only if f lies in the SONC cone.*

From the perspective of computation, computing SONC decompositions encounters the potential obstacle of enumerating exponentially many circuits since the number of lattice points contained in the Newton polytope grows exponentially with the number of variables and the polynomial degree. In order to develop efficient algorithms for certifying nonnegativity and polynomial optimization based on SONC decompositions, a core issue that must be addressed is, Which circuits are really needed when one seeks a SONC decomposition for a given polynomial? As the second contribution of this paper, we clarify an important fact that every SONC can decompose into a SONC polynomial by merely using the support of the original polynomial. In other words, SONC decompositions preserve the sparsity of polynomials. Actually, more is true. We prove that every SONC admits a SONC decomposition without cancellation via a connection with sums of binomial squares (SBS) (Theorem 1.3). This is dramatically different from the SOS decomposition of nonnegative polynomials, for which extra support and cancellation are needed in general.

Theorem 1.3. *If a polynomial f lies in the SONC cone, then f decompose into a SONC without cancellation.*

Theorem 1.3 provides a significant step toward bypassing the bottleneck of enumerating all circuits in the computation of SONC decompositions. In fact, this result also implies that the complexity of SONC/SAGE certificates does not depend on the polynomial degree, a sharp contrast with SOS-based certificates.

The rest of this paper is organized as follows. In section 2, we recall some basic facts about SONC. After that we consider the problem of which types of nonnegative polynomials lie in the SONC cone. We deal with the case of nonnegative polynomials with one negative term in section 3 and deal with the case of general nonnegative polynomials in section 4. In section 5, we prove that every SONC decomposes into a SONC polynomial without cancellation. Conclusions and discussions are given is section 6.

2. Preliminaries.

2.1. Notation and nonnegative polynomials.

Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ be the ring of real n-variate polynomials. Let $\mathbb{R}^*$ be the set of nonzero real numbers, $\mathbb{R}_{>0}$ the set of positive real numbers, and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. We use boldface to indicate a (column) vector, e.g., $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]^\intercal$. For a finite set $\mathscr{A} \subseteq \mathbb{N}^n$, we denote by $\operatorname{cone}(\mathscr{A})$ the conic hull of $\mathscr{A}$, by $\operatorname{conv}(\mathscr{A})$ the convex hull of $\mathscr{A}$, and by $V(\mathscr{A})$ the vertices of the convex hull of $\mathscr{A}$. We also denote by $V(P)$ the vertex set of a polytope P. We consider a polynomial $f \in \mathbb{R}[\mathbf{x}]$ supported on a finite set $\mathscr{A} \subseteq \mathbb{N}^n$, i.e., f is of the form $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}, \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *support* of f is $\operatorname{supp}(f) := \{\boldsymbol{\alpha} \in \mathscr{A} \mid c_{\boldsymbol{\alpha}} \neq 0\}$ and the *Newton polytope* of f is defined as $\operatorname{New}(f) := \operatorname{conv}(\operatorname{supp}(f))$. For a polytope P, we use P° to denote the relative interior of P. For a positive integer m, let $[m] := \{1, \ldots, m\}$.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ which is nonnegative over $\mathbb{R}^n$ is called a *nonnegative polynomial*. The class of nonnegative polynomials is denoted by PSD, which forms a closed convex cone.

A nonnegative polynomial must satisfy the following necessary conditions.

Proposition 2.1 (see [18, Theorem 3.6]). *Let $\mathscr{A} \subseteq \mathbb{N}^n$ and $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with $\operatorname{supp}(f) = \mathscr{A}$. Then f is nonnegative only if the following hold:*

1. $V(\mathscr{A}) \subseteq (2\mathbb{N})^n$.
2. *If* $\boldsymbol{\alpha} \in V(\mathscr{A})$, *then the corresponding coefficient* $c_{\boldsymbol{\alpha}}$ *is positive.*

For the remainder of this paper, we assume for simplicity that the monomial factor of any polynomial f is 1, that is, if $f = \mathbf{x}^{\boldsymbol{\alpha}'}(\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}})$ such that $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ and $\boldsymbol{\alpha}' \in \mathbb{N}^n$, then $\mathbf{x}^{\boldsymbol{\alpha}'} = 1$. Otherwise, we can always factor out the monomial factor.

2.2. Circuit polynomials. Following [18], a subset $\mathscr{A} \subseteq (2\mathbb{N})^n$ is called a *trellis* if $\mathscr{A}$ comprises the vertices of a simplex.

Definition 2.2 (see [5]). *Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a* circuit polynomial *if it is of the form*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \tag{2.1}$$

with $c_{\boldsymbol{\alpha}} \in \mathbb{R}_{>0}$ and $\boldsymbol{\beta} \in \operatorname{conv}(\mathscr{A})^{\circ}$. The support of a circuit polynomial is called a circuit.

Example 2.3. The Motzkin polynomial $f = 1+x^4y^2+x^2y^4-3x^2y^2$ is a nonnegative circuit polynomial.

For a circuit polynomial $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}$, since $\boldsymbol{\beta} \in \operatorname{conv}(\mathscr{A})^{\circ}$, $\boldsymbol{\beta}$ admits a unique convex representation: $\boldsymbol{\beta} = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}\boldsymbol{\alpha}$ with $\lambda_{\boldsymbol{\alpha}} > 0$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}} = 1$. Then we define the corresponding *circuit number* as $\Theta_f := \prod_{\boldsymbol{\alpha}\in\mathscr{A}}(c_{\boldsymbol{\alpha}}/\lambda_{\boldsymbol{\alpha}})^{\lambda_{\boldsymbol{\alpha}}}$. It is known that the nonnegativity of a circuit polynomial is decided by its circuit number alone.

Theorem 2.4 (see [5, Theorem 3.8]). *Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial and Θ_f its circuit number. Then f is nonnegative if and only if either $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le \Theta_f$ or $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $|d| \le \Theta_f$.*

Remark 2.5. We also view a monomial square (i.e., $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \in (2\mathbb{N})^n$) as a nonnegative circuit polynomial.

The following proposition characterizes the zeros of a circuit polynomial when the Newton polytope is full-dimensional.

Proposition 2.6 (see [5, Proposition 3.4 and Corollary 3.9]). *Let $f = \sum_{i=0}^n c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \Theta_f\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial, Θ_f the circuit number, and $\boldsymbol{\beta} = \sum_{i=0}^n \lambda_i\boldsymbol{\alpha}_i$ with $\lambda_i > 0$ and $\sum_{i=0}^n \lambda_i = 1$. Then f has exactly one zero $\mathbf{x}_*$ in $\mathbb{R}^n_{>0}$ which satisfies*

$$\frac{c_0\mathbf{x}_*^{\boldsymbol{\alpha}_0}}{\lambda_0} = \cdots = \frac{c_n\mathbf{x}_*^{\boldsymbol{\alpha}_n}}{\lambda_n} = \Theta_f\mathbf{x}_*^{\boldsymbol{\beta}}. \tag{2.2}$$

Moreover, if $\mathbf{x}$ is any zero of f, then $|\mathbf{x}| = \mathbf{x}_*$, i.e., $|x_i| = (x_*)_i$ for $i = 1, \dots, n$.

Proof. Consider $f' = \lambda_0 f/(c_0\mathbf{x}^{\boldsymbol{\alpha}_0})$. One can see that the zeros in $\mathbb{R}^n_{>0}$ of f coincide with the zeros in $\mathbb{R}^n_{>0}$ of f'. By Proposition 3.4 in [5], f' and hence f have exactly one zero $\mathbf{x}_*$ in $\mathbb{R}^n_{>0}$ which satisfies $\mathbf{x}_*^{\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_0} = (\lambda_i c_0)/(c_i\lambda_0)$ for $i = 1, \dots, n$. Let $s = (c_0\mathbf{x}_*^{\boldsymbol{\alpha}_0})/\lambda_0 = \cdots = (c_n\mathbf{x}_*^{\boldsymbol{\alpha}_n})/\lambda_n$. Then $s = \sum_{i=0}^n \lambda_i s = \sum_{i=0}^n c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i} = \Theta_f\mathbf{x}_*^{\boldsymbol{\beta}}$ and so (2.2) is proved. The last statement of the theorem follows from Corollary 3.9 in [5]. ∎

Remark 2.7. Note that in Proposition 2.6, $\mathbf{x}_* \in \mathbb{R}^n_{>0}$ and the circuit number Θ_f are uniquely determined by (2.2).

We shall say that a polynomial is a SONC *polynomial* if it is a SONC polynomial. For a nonnegative polynomial, an explicit representation as a SONC polynomial provides a certificate of its nonnegativity, which is called a *SONC decomposition*. The set of SONCs forms a closed convex cone called the *SONC cone*.

The following theorem from [5] adapted to our notation gives a characterization for a nonnegative polynomial to be a SONC when the Newton polytope is a simplex.

Theorem 2.8 (see [5, Corollary 7.5]). *Let* $f = \sum_{i=0}^{n} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ *be nonnegative with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 0, \ldots, n$ *such that* $\mathrm{New}(f)$ *is a simplex and* $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ$ *for* $j = 1, \ldots, l$. *If there exists a point* $\boldsymbol{v} = [v_k] \in (\mathbb{R}^*)^n$ *such that* $d_j \boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ *for all* j, *then* f *lies in the SONC cone.*

3. Nonnegative polynomials with one negative term. Following the line of Theorem 2.8, we now study which types of nonnegative polynomials with general Newton polytopes lie in the SONC cone. The well-known Hilbert's classification on the coincidence of nonnegative polynomials and sums of squares is according to the number of variables and the degree of polynomials. We will see that the related classification for SONCs depends on the combinatorical structure of supports of polynomials. In this section, we deal with the case of nonnegative polynomials with one negative term (by a negative term we refer to a term that takes a negative value at some point), i.e., polynomials of the form $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \ldots, m$ and $\boldsymbol{\beta} \notin V(\mathrm{New}(f_d))$. Let $\partial\mathrm{New}(f_d)$ denote the boundary of $\mathrm{New}(f_d)$. We first reduce the case of $\boldsymbol{\beta} \in \partial\mathrm{New}(f_d)$ to the case $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$ by the following lemma.

Lemma 3.1. *Let* $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ *with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \ldots, m$, *and* $\boldsymbol{\beta} \in \partial\mathrm{New}(f_d)$. *Furthermore, let* F *be the minimal face of* $\mathrm{New}(f_d)$ *containing* $\boldsymbol{\beta}$. *Then* f_d *is nonnegative if and only if the restriction of* f_d *to the face* F *is nonnegative.*

Proof. The necessity follows from [18, Theorem 3.6]. For the sufficiency, note that the restriction to the face F contains the term $-d\mathbf{x}^{\boldsymbol{\beta}}$ and this restriction is nonnegative. Moreover, all other terms in f_d are monomial squares. Hence f_d is nonnegative. ∎

From now on, we assume $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$. Without loss of generality, we further make the assumption that the Newton polytope of f_d is full-dimensional, i.e., $\dim(\mathrm{New}(f_d)) = n$. Otherwise, we can reduce to this case by applying an appropriate monomial transformation to f_d [13].

To begin with, we give a characterization for f_d to be nonnegative as well as the positive zeros of f_d in a similar manner as Theorem 2.4 and Proposition 2.6. It turns out that f_d behaves just like a circuit polynomial.

It is not hard to see that the set $\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}$ is nonempty and is bounded from above. So it has a supremum. Let

$$d^* \triangleq \sup\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}. \tag{3.1}$$

The quantity d^* is an analogue of the circuit number for f_d.

Proposition 3.2. *Let* $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ *with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \ldots, m$ *such that* $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$, $\dim(\mathrm{New}(f_d)) = n$, *and let* d^* *be defined as (3.1). Then* f_d *is* *nonnegative if and only if either $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le d^*$ or $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $|d| \le d^*$. Moreover, f_{d^*} has exactly one zero in $\mathbb{R}^n_{>0}$.*

Proof. First, if $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le 0$, then f_d is obviously nonnegative since it is a sum of monomial squares. If $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $d \le 0$, then f_d is nonnegative if and only if f_{-d} is nonnegative. Thus without loss of generality, we may assume $d > 0$. Since the only negative term of f_d is $-d\mathbf{x}^{\boldsymbol{\beta}}$, f_d is nonnegative over $\mathbb{R}^n$ if and only if f_d is nonnegative over $\mathbb{R}^n_{>0}$. Therefore, by the definition of d^*, f_d is nonnegative if and only if $d \le d^*$.

To prove the second statement, let us consider $f' = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}}$. It is not hard to see $d^* = \inf_{\mathbf{x} \in \mathbb{R}^n_{>0}} f'$. Because $\dim(\mathrm{New}(f_d)) = n$ and $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$, we have $\dim(\mathrm{conv}(\{\boldsymbol{\alpha}_1 - \boldsymbol{\beta}, \dots, \boldsymbol{\alpha}_m - \boldsymbol{\beta}\})) = n$ and $\mathbf{0} \in \mathrm{cone}(\{\boldsymbol{\alpha}_1 - \boldsymbol{\beta}, \dots, \boldsymbol{\alpha}_m - \boldsymbol{\beta}\})^\circ$. Therefore, by Theorem 3.4 in [13], f' attains its minimum over $\mathbb{R}^n_{>0}$ at a unique minimizer. Since the minimizers of f' over $\mathbb{R}^n_{>0}$ coincide with the zeros of f_{d^*} in $\mathbb{R}^n_{>0}$, it follows that f_{d^*} has exactly one zero in $\mathbb{R}^n_{>0}$. ∎

For a nonnegative polynomial $f_d = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$, let $\mathscr{C}$ be the set of all circuits $\mathscr{T} \cup \{\boldsymbol{\beta}\}$ with $\mathscr{T} \subseteq \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}$. In the rest of this section, we will prove that f_d decomposes into a SONC polynomial that are supported on circuits in $\mathscr{C}$. We first consider the decomposition of f_{d^*} and then get the decomposition of f_d from that of f_{d^*}. By using undetermined coefficients, the existence of such a decomposition of f_{d^*} is reduced to the existence of a nonnegative solution for a particular linear system, which can be further reduced to the existence of a nonnegative solution for a tuple of subsystems by virtue of the following result, known as Helly's theorem.

Theorem 3.3 (Helly [4]). *Let $X_1, \dots, X_r$ be a finite collection of convex subsets of $\mathbb{R}^s$ with $r > s$. If the intersection of every $s+1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Next, using Helly's theorem, we prove a result concerning the existence of nonnegative solutions to a particular class of linear systems for later use, which might be also of independent interest. To state the result, we need the following notation. Let $A = [a_{ij}] \in \mathbb{R}^{m \times r}$, $\mathbf{b} = [b_i] \in \mathbb{R}^m$, and $\mathbf{z} = (z_1, \dots, z_r)^\intercal$ be a set of variables. Then for each $j \in [r]$, we write $\bar{\mathbf{z}}_j := \mathbf{z} \setminus z_j$, $\bar{\mathbf{b}}_j := [b_i]_{i \text{ with } a_{ij}=0}$ (namely, removing the entries with $a_{ij} \neq 0$ from $\mathbf{b}$) and denote by A_j the submatrix of A by deleting all of the ith rows with $a_{ij} \neq 0$ and the jth column from A such that $A_j \bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ is the subsystem of $A\mathbf{z} = \mathbf{b}$ after removing the equations involving the variable z_j.

Lemma 3.4. *Let $A = [a_{ij}] \in \mathbb{R}^{m \times r}$, $\mathbf{b} = [b_i] \in \mathbb{R}^m$, and $\mathbf{z} = (z_1, \dots, z_r)^\intercal$ be a set of variables. Assume that $A\mathbf{z} = \mathbf{b}$ is consistent, $\mathrm{rank}(A) > 1$, and $\mathrm{rank}(A_j) = \mathrm{rank}(A) - 1$ for all $j \in [r]$. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if $A_j \bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution for $j = 1, \dots, r$.*

Proof. Let $t = \mathrm{rank}(A) > 1$. Then the system of linear equations $A\mathbf{z} = \mathbf{b}$ has $r - t$ free variables. Without loss of generality, let the $r - t$ free variables be $\{z_1, \dots, z_{r-t}\}$ so that we can solve for $\{z_{r-t+1}, \dots, z_r\}$ from $A\mathbf{z} = \mathbf{b}$ to obtain $z_i = h_i(z_1, \dots, z_{r-t})$ with h_i being a linear function, $i = r - t + 1, \dots, r$. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if the set

$$\begin{aligned}\{(z_1,\ldots,z_{r-t})\in\mathbb{R}^{r-t}\ |z_i\ge 0 \text{ for } i\in[r-t] \text{ and}\\ z_i=h_i(z_1,\ldots,z_{r-t})\ge 0 \text{ for } i\in\{r-t+1,\ldots,r\}\}\end{aligned}\tag{3.2}$$

is nonempty. Define $X_i \coloneqq \{(z_1,\ldots,z_{r-t})\in\mathbb{R}^{r-t}\mid z_i\ge 0\}$ for $i=1,\ldots,r-t$ and $X_i \coloneqq \{(z_1,\ldots,z_{r-t})\in\mathbb{R}^{r-t}\mid h_i(z_1,\ldots,z_{r-t})\ge 0\}$ for $i=r-t+1,\ldots,r$, which are all convex subsets of $\mathbb{R}^{r-t}$. Therefore, by Theorem 3.3, the intersection of all X_i, i.e., (3.2), is nonempty if and only if the intersection of every $r-t+1$ of these sets is nonempty. Because of $t>1$, the latter is in turn equivalent to the fact that the intersection of every $r-1$ of these sets is nonempty, that is, the set

$$\begin{aligned}\{(z_1,\ldots,z_{r-t})\in\mathbb{R}^{r-t}\ |z_i\ge 0 \text{ for } i\in[r-t]\setminus\{j\} \text{ and}\\ z_i=h_i(z_1,\ldots,z_{r-t})\ge 0 \text{ for } i\in\{r-t+1,\ldots,r\}\}\end{aligned}\tag{3.3}$$

is nonempty for $j=1,\ldots,r-t$ and the set

$$\begin{aligned}\{(z_1,\ldots,z_{r-t})\in\mathbb{R}^{r-t}\ |z_i\ge 0 \text{ for } i\in[r-t] \text{ and}\\ z_i=h_i(z_1,\ldots,z_{r-t})\ge 0 \text{ for } i\in\{r-t+1,\ldots,r\}\setminus\{j\}\}\end{aligned}\tag{3.4}$$

is nonempty for $j=r-t+1,\ldots,r$.

For $j=1,\ldots,r-t$, (3.3) is nonempty if and only if $A\mathbf{z}=\mathbf{b}$ has a solution with $\bar{\mathbf{z}}_j\in\mathbb{R}^{r-1}_{\ge 0}$ and $z_j\in\mathbb{R}$, which is equivalent to the condition that $A_j\bar{\mathbf{z}}_j=\bar{\mathbf{b}}_j$ has a nonnegative solution since $\operatorname{rank}(A_j)=\operatorname{rank}(A)-1$. For $j=r-t+1,\ldots,r$, (3.4) is nonempty if and only if $A\mathbf{z}=\mathbf{b}$ has a solution with $\bar{\mathbf{z}}_j\in\mathbb{R}^{r-1}_{\ge 0}$ and $z_j\in\mathbb{R}$, which is also equivalent to the condition that $A_j\bar{\mathbf{z}}_j=\bar{\mathbf{b}}_j$ has a nonnegative solution since $\operatorname{rank}(A_j)=\operatorname{rank}(A)-1$. Put all of the above together and we deduce that $A\mathbf{z}=\mathbf{b}$ has a nonnegative solution if and only if $A_j\bar{\mathbf{z}}_j=\bar{\mathbf{b}}_j$ has a nonnegative solution for $j=1,\ldots,r$ as desired. ■

Example 3.5. Consider the linear system $S=\{z_1+z_2=1, z_3+z_4=2, z_2+z_3=1, z_1+z_4=2, z_1+z_2+z_3+z_4=3\}$. One can check that S satisfies the hypotheses of Lemma 3.4 with $r=4, t=3$. We see at once that all subsystems $\{z_3+z_4=2, z_2+z_3=1\}, \{z_3+z_4=2, z_1+z_4=2\}, \{z_1+z_2=1, z_1+z_4=2\}, \{z_1+z_2=1, z_2+z_3=1\}$ admit a nonnegative solution. Thus by Lemma 3.4 we conclude that S has a nonnegative solution.

Lemma 3.4 assumes the consistency of $A\mathbf{z}=\mathbf{b}$. It is known that the system of linear equations $A\mathbf{z}=\mathbf{b}$ is consistent if and only if $\mathbf{b}$ belongs to the image of A. For later use, we give a more concrete description concerning the consistency of $A\mathbf{z}=\mathbf{b}$ here, whose correctness is obvious, and thus we omit the proof.

Lemma 3.6. *Let $A=[a_{ij}]\in\mathbb{R}^{m\times r}$, $\mathbf{b}=[b_j]\in\mathbb{R}^m$, and $\mathbf{z}=(z_1,\ldots,z_r)^\intercal$ be a set of variables. Assume that the row vectors of the matrix C span the cokernel of A (i.e., the left null space of A). Then $A\mathbf{z}=\mathbf{b}$ is consistent if and only if $C\mathbf{b}=\mathbf{0}$.*

Now we are ready to prove that f_{d^*} lies in the SONC cone.

Lemma 3.7. *Let $f_d=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-d\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i\in(2\mathbb{N})^n, c_i\in\mathbb{R}_{>0}, i=1,\ldots,m$, such that $\boldsymbol{\beta}\in\operatorname{New}(f_d)^\circ$, $\dim(\operatorname{New}(f_d))=n$, and let d^* be defined as (3.1). Then f_{d^*} lies in the SONC cone.*

Proof. If $m = n+1$, then f_d is a circuit polynomial and of course f_{d^*} lies in the SONC cone. Assume now $m > n+1$. By Proposition 3.2, f_{d^*} has exactly one zero in $\mathbb{R}^n_{>0}$, which is denoted by $\mathbf{x}_*$. Let

$$\{\Delta_1, \dots, \Delta_r\} \coloneqq \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta} \in \Delta^\circ, V(\Delta) \subseteq \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}\}$$

and $I_k \coloneqq \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_k)\}$ for $k = 1, \dots, r$. We complete the proof by constructing a SONC decomposition supported on the simplices $\{\Delta_k\}_k$ for f_{d^*}.

First, we assume $\dim(\Delta_k) = n$ so that $|I_k| = n+1$ for $k = 1, \dots, r$. For each Δ_k, since $\boldsymbol{\beta} \in \Delta_k^\circ$, we can write $\boldsymbol{\beta} = \sum_{i \in I_k} \lambda_{ik} \boldsymbol{\alpha}_i$, where $\sum_{i \in I_k} \lambda_{ik} = 1, \lambda_{ik} > 0, i \in I_k$. Inspired by Proposition 2.6 and using undetermined coefficients, we may consider the following system of linear equations in variables $\{c_{ik}\}_{i,k}$ and $\{s_k\}_k$:

$$\begin{cases} \dfrac{c_{ik}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ik}} = s_k & \text{for } i \in I_k, k = 1, \dots, r, \\ \displaystyle\sum_{i \in I_k} c_{ik} = c_i & \text{for } i = 1, \dots, m. \end{cases} \tag{3.5}$$

Eliminate the variables $\{c_{ik}\}_{i,k}$ from (3.5) and we obtain

$$\sum_{i \in I_k} \lambda_{ik} s_k = c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} \quad \text{for } i = 1, \dots, m. \tag{3.6}$$

If (3.6) has a nonnegative solution, then we can retrieve a SONC decomposition for f_{d^*} from this nonnegative solution as follows. Assume that $\{s_1^*, \dots, s_r^*\}$ is a nonnegative solution to (3.6). Substitute $\{s_1^*, \dots, s_r^*\}$ into the system of equations (3.5), and we have $c_{ik} = \lambda_{ik} s_k^* / \mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i \in I_k, k = 1, \dots, r$. Let $d_k = s_k^* / \mathbf{x}_*^{\boldsymbol{\beta}}$ and $f_k = \sum_{i \in I_k} c_{ik} \mathbf{x}^{\boldsymbol{\alpha}_i} - d_k \mathbf{x}^{\boldsymbol{\beta}}$ for $k = 1, \dots, r$. Then by (3.5) and by Proposition 2.6, d_k is the circuit number of f_k and hence f_k is a nonnegative circuit polynomial for all k. By (3.5), $\sum_{k=1}^r d_k \mathbf{x}_*^{\boldsymbol{\beta}} = \sum_{k=1}^r \sum_{i \in I_k} c_{ik} \mathbf{x}_*^{\boldsymbol{\alpha}_i} = \sum_{i=1}^m c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} = d^* \mathbf{x}_*^{\boldsymbol{\beta}}$, which implies $\sum_{k=1}^r d_k = d^*$. It follows that $f_{d^*} = \sum_{k=1}^r f_k$ lies in the SONC cone as desired. So our remaining task is to prove that (3.6) has a nonnegative solution.

Claim. The linear system (3.6) in variables $\{s_1, \dots, s_r\}$ has a nonnegative solution.

Proof of the claim. Denote the coefficient matrix of (3.6) by $A = [a_{ik}] \in \mathbb{R}^{m \times r}$ (satisfying $a_{ik} = \lambda_{ik}$ if $i \in I_k$ and $a_{ik} = 0$ otherwise) and denote the coefficient matrix of

$$\sum_{i \in I_k} \lambda_{ik} s_k = c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} \quad \text{for } i \in [m] \setminus I_j \tag{3.7}$$

by A_j for each $j \in [r]$. Note that (3.7) is obtained from (3.6) by removing the equations involving the variable s_j. In order to invoke Lemma 3.4 to prove that (3.6) has a nonnegative solution, we need to check the following hypotheses:

1. $\operatorname{rank}(A) > 1$;
2. $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$ for each $j \in [r]$;
3. (3.6) is consistent.

Fix $j \in [r]$. For every $i \in [m] \setminus I_j$, since $\boldsymbol{\beta} \in \Delta_j^\circ$, there exists a facet F of Δ_j such that $\boldsymbol{\beta} \in \operatorname{conv}(V(F) \cup \{\boldsymbol{\alpha}_i\})^\circ$. Let $\operatorname{conv}(V(F) \cup \{\boldsymbol{\alpha}_i\}) = \Delta_{p_i}$ for some $p_i \in [r]$ (see Figure 1). It is

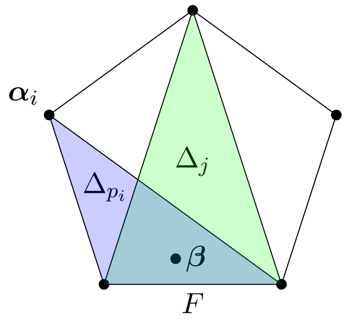


Figure 1. *Illustration for the correspondence between* $\boldsymbol{\alpha}_i$ *and* Δ_{p_i} *for* $i \in [m] \setminus I_j$

not hard to see $p_{i_1} \neq p_{i_2}$ whenever $i_1 \neq i_2$. For every $k \in [r] \setminus (\{j\} \cup \{p_i \mid i \in [m] \setminus I_j\})$, let $s_k = 0$ in (3.7) and then by construction we obtain

$$\lambda_{ip_i} s_{p_i} = c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} \quad \text{for } i \in [m] \setminus I_j. \tag{3.8}$$

It follows that $\operatorname{rank}(A_j) = m - |I_j| = m - (n+1)$ and furthermore, $\operatorname{rank}(A) \geq \operatorname{rank}(A_j) + 1 = m - n$, $\dim(\operatorname{coker}(A)) = m - \operatorname{rank}(A) \leq n$. Let $C := [\boldsymbol{\alpha}_1 - \boldsymbol{\beta}, \ldots, \boldsymbol{\alpha}_m - \boldsymbol{\beta}]$. Then,

$$\begin{aligned} CA &= \left[\sum_{i=1}^m (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) a_{i1}, \ldots, \sum_{i=1}^m (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) a_{ir}\right] \\ &= \left[\sum_{i \in I_1} (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \lambda_{i1}, \ldots, \sum_{i \in I_r} (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \lambda_{ir}\right] \\ &= \left[\sum_{i \in I_1} \lambda_{i1} \boldsymbol{\alpha}_i - \boldsymbol{\beta}, \ldots, \sum_{i \in I_r} \lambda_{ir} \boldsymbol{\alpha}_i - \boldsymbol{\beta}\right] = [\mathbf{0}, \ldots, \mathbf{0}]. \end{aligned}$$

So the row vectors of C belong to the cokernel of A. We have $\operatorname{rank}(C) = \operatorname{rank}(\{\boldsymbol{\alpha}_i - \boldsymbol{\beta}\}_{i=1}^m) = n$ because $\boldsymbol{\beta} \in \operatorname{New}(f_d)^\circ$ and $\dim(\operatorname{New}(f_d)) = n$. As $\dim(\operatorname{coker}(A)) \leq n$, we then conclude that $\dim(\operatorname{coker}(A)) = n$ and the row vectors of C span the cokernel of A. As a result, $\operatorname{rank}(A) = m - n > 1$ and $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$. Because the zero $\mathbf{x}_*$ is also a minimizer of f_{d^*}, it satisfies $\{f_{d^*}(\mathbf{x}_*) = 0, \nabla f_{d^*}(\mathbf{x}_*) = \mathbf{0}\}$ (∇ denotes the gradient with respect to $\mathbf{x}$) which gives

$$\begin{cases} \displaystyle\sum_{i=1}^m c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} - d^* \mathbf{x}_*^{\boldsymbol{\beta}} = 0, \\ \displaystyle\sum_{i=1}^m c_i \boldsymbol{\alpha}_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} - d^* \boldsymbol{\beta} \mathbf{x}_*^{\boldsymbol{\beta}} = \mathbf{0}. \end{cases} \tag{3.9}$$

It follows that $\sum_{i=1}^m c_i (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \mathbf{x}_*^{\boldsymbol{\alpha}_i} = \mathbf{0}$, i.e., $C \cdot [c_1 \mathbf{x}_*^{\boldsymbol{\alpha}_1}, \ldots, c_m \mathbf{x}_*^{\boldsymbol{\alpha}_m}]^\intercal = \mathbf{0}$. Thus by Lemma 3.6, (3.6) is consistent.

Now by Lemma 3.4, in order to prove the claim, we only need to show that every subsystem (3.7) in variables $\{s_1,\dots,s_r\}\setminus\{s_j\}$ has a nonnegative solution for $j=1,\dots,r$. Given $j\in[r]$, from (3.8) we have $s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{ip_i}$ for $i\in[m]\setminus I_j$. Hence

$$\begin{cases} s_k=0, & \text{for } k\in[r]\setminus(\{j\}\cup\{p_i\mid i\in[m]\setminus I_j\}),\\ s_{p_i}=\dfrac{c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ip_i}} & \text{for } i\in[m]\setminus I_j \end{cases} \tag{3.10}$$

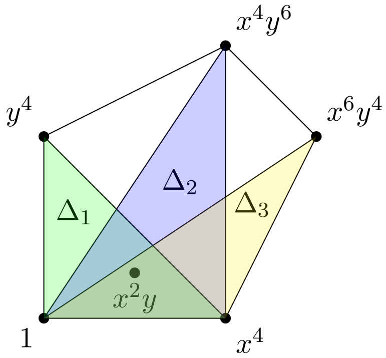
is a nonnegative solution to (3.7). So the claim is proved.

For the case that $\dim(\Delta_k)=n$ does not hold for all k, note that all of the above results remain valid for $\boldsymbol{\beta}\in\mathbb{R}^n$. We then give $\boldsymbol{\beta}$ a small perturbation, say, $\boldsymbol{\delta}$, such that $\dim(\Delta_k)=n$ holds for all k. Then the new linear system (3.6) for $\boldsymbol{\beta}+\boldsymbol{\delta}$ has a nonnegative solution. Let $\boldsymbol{\delta}\to\mathbf{0}$. We obtain that (3.6) also has a nonnegative solution for $\boldsymbol{\beta}$. Thus the theorem remains true in this case. ■

We give an example to illustrate Lemma 3.7.

Example 3.8. Let $f_d=1+x^4+y^4+x^6y^4+x^4y^6-dx^2y$ and $d^*=\sup\{d\in\mathbb{R}_{>0}\mid f_d \text{ is nonnegative}\}$. We have

$$\begin{bmatrix}2\\1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}4\\0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}0\\4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{3}\begin{bmatrix}4\\0\end{bmatrix}+\frac{1}{6}\begin{bmatrix}4\\6\end{bmatrix}=\frac{5}{8}\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{8}\begin{bmatrix}4\\0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}6\\4\end{bmatrix}.$$



The system of equations $\{f_d=0,\nabla f_d=\mathbf{0}\}$ in variables $\{x,y,d\}$ has exactly one zero ($x_*\approx 0.944112, y_*\approx 0.708568, d^*\approx 3.682248$) in $\mathbb{R}^3_{>0}$. The linear system (3.6) becomes

$$\begin{cases} \frac{1}{4}s_1+\frac{1}{2}s_2+\frac{5}{8}s_3=1,\\ \frac{1}{2}s_1+\frac{1}{3}s_2+\frac{1}{8}s_3=x_*^4,\\ \frac{1}{4}s_1=y_*^4,\\ \frac{1}{4}s_3=x_*^6y_*^4,\\ \frac{1}{6}s_2=x_*^4y_*^6, \end{cases} \tag{3.11}$$

which has a nonnegative solution $(s_1\approx 1.00829, s_2\approx 0.603299, s_3\approx 0.714045)$. Thus from the proof of Lemma 3.7, we obtain a SONC decomposition of f_{d^*}, which is

$$\begin{aligned} f_{d^*} \approx\ & (0.252072 + 0.634543x^4 + y^4 - 1.59646x^2y) \\ & + (0.30165 + 0.253115x^4 + x^4y^6 - 0.955222x^2y) \\ & + (0.446278 + 0.112342x^4 + x^6y^4 - 1.13057x^2y). \end{aligned}$$

Theorem 3.9. *Let* $f_d = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ *with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \dots, m$ *such that* $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ$, $\dim(\mathrm{New}(f_d)) = n$. *Then* f_d *is nonnegative if and only if* f_d *lies in the SONC cone.*

Proof. The sufficiency is obvious. For the necessity, assume that f_d is nonnegative. If $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d < 0$, or $d = 0$, then f_d is a sum of monomial squares and so f_d lies in the SONC cone. If $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $d < 0$, through a variable transformation $x_j \mapsto -x_j$ for some odd number β_j, we can always assume $d > 0$. Let d^* be defined as (3.1). By Lemma 3.7 and its proof, f_{d^*} lies in the SONC cone and f_{d^*} admits a SONC decomposition: $f_{d^*} = \sum_{k=1}^r(\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial with d_k being the corresponding circuit number for all k (the sets $I_k, k \in [r]$ are defined in the proof of Lemma 3.7). Since f_d is nonnegative, it follows that $d \le d^*$. We have $f_d = \sum_{k=1}^r(\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial for all k by Theorem 2.4. Thus f_d lies in the SONC cone. ■

Remark 3.10. Theorem 3.9 is a generalization of Theorem 2.8 to the case of polynomials with general Newton polytopes and with a unique negative term. We point out that a special case of Theorem 3.9 concerning agiforms was proved by Reznick in 1989; see [18, Theorem 7.1].

Definition 3.11 (see [9]). *An* AGE polynomial *is a nonnegative polynomial with at most one negative term, namely, it is nonnegative and of the form*

$$\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}, \text{ where } \boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_+, i = 1, \dots, m,$$

and either $\boldsymbol{\beta} \in \mathbb{N}^n \setminus (2\mathbb{N})^n$ *or* $\boldsymbol{\beta} \in (2\mathbb{N})^n$ *and* $d \ge 0$.

The proof of Theorem 3.9 enables us to give a SONC decomposition *without cancellation* for AGE polynomials.

Theorem 3.12. *Let* $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ *with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \dots, m$ *be an AGE polynomial. Let*

$$\mathscr{F} \coloneqq \{\Delta \mid \Delta \text{ is a simplex}, \boldsymbol{\beta} \in \Delta^\circ, V(\Delta) \subseteq \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}\}.$$

Then f *admits a SONC decomposition:*

$$f = \sum_{\Delta\in\mathscr{F}} f_\Delta + \sum_{i\in I} c_i\mathbf{x}^{\boldsymbol{\alpha}_i}, \tag{3.12}$$

where f_Δ *is a nonnegative circuit polynomial supported on* $V(\Delta) \cup \{\boldsymbol{\beta}\}$ *for each* Δ *and* $I = \{i \in [m] \mid \boldsymbol{\alpha}_i \notin \bigcup_{\Delta\in\mathscr{F}} V(\Delta)\}$.

Proof. It follows from Lemma 3.1 and the proof of Theorem 3.9. ∎

Murray, Chandrasekaran, and Wiermann proposed SAGE polynomials as a new nonnegativity certificate of polynomials in [9], where they considered not only polynomial nonnegativity but also signomial nonnegativity. Nonnegative polynomials that admit a SAGE decomposition are called *SAGE polynomials*. The cone containing all SAGE polynomials is called the *SAGE cone*. Due to Theorem 3.12, we immediately obtain the following result.

Corollary 3.13. *The SONC cone coincides with the SAGE cone.*

The coincidence of the SONC cone and the SAGE cone was also independently proved in [9] by showing that any extreme ray of the SAGE cone is a nonnegative circuit polynomial [9, Corollary 21]. The proof in [9] was provided in the context of signomials and stems from convex duality. In contrast, our proof uses algebraic techniques and exploits the combinatorical structure of the polynomial support in an essential way.

4. Nonnegative polynomials with multiple negative terms. In this section, we treat arbitrary nonnegative polynomials, not just those with at most one negative term, and provide sufficient conditions under which an arbitrary nonnegative polynomial admits a SONC decomposition. The proof proceeds in a similar manner as that of Theorem 3.9. We first consider the case that the polynomial lies on the boundary of the PSD cone since the general case will be reduced to this case. As in the proof of Lemma 3.7, by using undetermined coefficients, the existence of such a decomposition is reduced to the existence of a nonnegative solution for a particular linear system, which is then further reduced to the existence of a nonnegative solution for a tuple of subsystems by Lemma 3.4.

To state the theorem, we need a technical condition on the Newton polytope of a polynomial. Let Δ be a polytope of dimension d. We say that a vertex $\boldsymbol{\alpha}$ of Δ is *simple* if $\boldsymbol{\alpha}$ lies on precisely d edges.

Theorem 4.1. *Let $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1,\dots,m$, $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ, j = 1,\dots,l$. Assume that some vertex of $\mathrm{New}(f)$ is simple, that all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ (in the affine subspace spanned by the support of f), and that there exists a point $\boldsymbol{v} = [v_k] \in (\mathbb{R}^*)^n$ such that $d_j\boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j. Then f is nonnegative if and only if f lies in the SONC cone.*

Proof. First assume $\dim(\mathrm{New}(f)) = n$ (so $m \geq n+1$). Otherwise, we can reduce to this case by applying an appropriate monomial transformation to f. If $l = 1$, then the conclusion follows from Theorem 3.9. We assume now $l > 1$. The sufficiency is obvious. For the necessity, suppose that f is nonnegative. After a variable transformation $x_k \mapsto -x_k$ for all k with $v_k < 0$, we can assume $d_j > 0$ for all j. Let

$$d_l^* \triangleq \sup\{\tilde{d}_l \in \mathbb{R} \mid \tilde{f} = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - \tilde{d}_l\mathbf{x}^{\boldsymbol{\beta}_l} \text{ is nonnegative}\}. \tag{4.1}$$

Note that d_l^* is well-defined since the set in (4.1) is nonempty and is bounded from above. Let $f^* = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - d_l^*\mathbf{x}^{\boldsymbol{\beta}_l}$. Then $f^* = 0$ has a zero in $\mathbb{R}_{>0}^n$ [22, Lemma 4.2], which is denoted by $\mathbf{x}_*$. The assumption that all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ implies that if a simplex Δ with vertices coming from $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ contains some $\boldsymbol{\beta}_j$, then $\dim(\Delta) = n$ and it contains all $\boldsymbol{\beta}_j$. Let

$$\{\Delta_1, \ldots, \Delta_r\} \coloneqq \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta}_j \in \Delta^\circ, j \in [l], V(\Delta) \subseteq \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}\}$$

and $I_k \coloneqq \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_k)\}$ for $k = 1, \ldots, r$. We have $\dim(\Delta_k) = n$ for all k. For every $\boldsymbol{\beta}_j$ and every Δ_k, since $\boldsymbol{\beta}_j \in \Delta_k^\circ$, we can write $\boldsymbol{\beta}_j = \sum_{i \in I_k} \lambda_{ijk} \boldsymbol{\alpha}_i$, where $\sum_{i \in I_k} \lambda_{ijk} = 1, \lambda_{ijk} > 0, i \in I_k$. In a similar manner as we proved Lemma 3.7, let us consider the following system of linear equations in variables $\{c_{ijk}\}_{i,j,k}$, $\{d_{jk}\}_{j,k}$ and $\{s_{jk}\}_{j,k}$:

$$(4.2) \qquad \begin{cases} \dfrac{c_{ijk}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ijk}} = d_{jk}\mathbf{x}_*^{\boldsymbol{\beta}_j} = s_{jk} & \text{for } i \in I_k, k = 1, \ldots, r, j = 1, \ldots, l, \\ \displaystyle\sum_{k=1}^{r} d_{jk} = d_j & \text{for } j = 1, \ldots, l-1, \\ \displaystyle\sum_{k=1}^{r} d_{lk} = d_l^* & \\ \displaystyle\sum_{j=1}^{l} \sum_{i \in I_k} c_{ijk} = c_i & \text{for } i = 1, \ldots, m. \end{cases}$$

Eliminate the variables $\{c_{ijk}\}_{i,j,k}$ and $\{d_{jk}\}_{j,k}$ from (4.2) and we obtain

$$(4.3) \qquad \begin{cases} \displaystyle\sum_{j=1}^{l} \sum_{i \in I_k} \lambda_{ijk} s_{jk} = c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i} & \text{for } i = 1, \ldots, m, \\ \displaystyle\sum_{k=1}^{r} s_{jk} = d_j \mathbf{x}_*^{\boldsymbol{\beta}_j} & \text{for } j = 1, \ldots, l-1, \\ \displaystyle\sum_{k=1}^{r} s_{lk} = d_l^* \mathbf{x}_*^{\boldsymbol{\beta}_l}. & \end{cases}$$

If (4.3) has a nonnegative solution, then we can retrieve a SONC decomposition supported on the simplices $\{\Delta_k\}_k$ for f^* as follows. Assume that $\{s_{jk}^*\}_{j,k}$ is a nonnegative solution to (4.3). Substitute $\{s_{jk}^*\}_{j,k}$ into the system of equations (4.2), and we have $c_{ijk} = \lambda_{ijk} s_{jk}^* / \mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i \in I_k, k = 1, \ldots, r, j = 1, \ldots, l$. Let $f_{jk} = \sum_{i \in I_k} c_{ijk} \mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk} \mathbf{x}^{\boldsymbol{\beta}_j}$ for $k = 1, \ldots, r, j = 1, \ldots, l-1$. Then by (4.2) and by Proposition 2.6, d_{jk} is the circuit number of f_{jk} and f_{jk} is a nonnegative circuit polynomial for all j, k. By (4.2), we have $f = \sum_{j=1}^{l-1} \sum_{k=1}^{r} f_{jk} + \sum_{k=1}^{r} (\sum_{i \in I_k} c_{ilk} \mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d_l}{d_l^*} d_{lk} \mathbf{x}^{\boldsymbol{\beta}_l})$. Since $d_l \le d_l^*$, $\sum_{i \in I_k} c_{ilk} \mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d_l}{d_l^*} d_{lk} \mathbf{x}^{\boldsymbol{\beta}_l}$ is a nonnegative circuit polynomial for all k by Theorem 2.4. Thus f lies in the SONC cone as desired. Our remaining task hence is to prove the following claim.

Claim. The linear system (4.3) in variables $\{s_{jk}\}_{j,k}$ has a nonnegative solution.

Proof of the claim. Denote the coefficient matrix of (4.3) by A and denote the coefficient matrix of

$$
\begin{cases}
\sum_{j=1}^{l}\sum_{i\in I_k}\lambda_{ijk}s_{jk}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i} & \text{for } i\in[m]\setminus I_v,\\
\sum_{k=1}^{r}s_{jk}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j} & \text{for } j\in[l-1]\setminus\{u\},\\
\sum_{k=1}^{r}s_{lk}=d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l} & \text{if } u\neq l
\end{cases}
\tag{4.4}
$$

by A_{uv} for every $u\in[l]$ and every $v\in[r]$. Note that (4.4) is obtained from (4.3) by removing the equations involving the variable s_{uv}. In order to invoke Lemma 3.4 to prove that (4.3) has a nonnegative solution, we need to check the following hypotheses:

1. $\mathrm{rank}(A)>1$;
2. $\mathrm{rank}(A_{uv})=\mathrm{rank}(A)-1$ for every $u\in[l]$ and every $v\in[r]$;
3. (4.3) is consistent.

Fix $u\in[l]$ and $v\in[r]$. For every $i\in[m]\setminus I_v$, since $\boldsymbol{\beta}_u\in\Delta_v^\circ$, there exists a facet F of Δ_v such that $\boldsymbol{\beta}_u\in\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})^\circ$. Let $\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})=\Delta_{p_i}$ for some $p_i\in[r]$. It holds that $p_{i_1}\neq p_{i_2}$ whenever $i_1\neq i_2$. For every pair (j,k) such that $j=u,k\in[r]\setminus(\{v\}\cup\{p_i\mid i\in[m]\setminus I_v\})$ or $j\in[l]\setminus\{u\},k\in[r]\setminus\{v\}$, let $s_{jk}=0$ in (4.4), and then we obtain

$$
\begin{cases}
\lambda_{iup_i}s_{up_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i} & \text{for } i\in[m]\setminus I_v,\\
s_{jv}=d_j\mathbf{x}_*^{\boldsymbol{\beta}_j} & \text{for } j\in[l-1]\setminus\{u\},\\
s_{lv}=d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l} & \text{if } u\neq l.
\end{cases}
\tag{4.5}
$$

It follows that A_{uv} has full rank and $\mathrm{rank}(A_{uv})=m-|I_v|+l-1=m+l-(n+2)$. Moreover, $\mathrm{rank}(A)\geq\mathrm{rank}(A_{uv})+1=m+l-(n+1)$ and $\dim(\mathrm{coker}(A))=m+l-\mathrm{rank}(A)\leq n+1$. Let $C:=\left[\begin{smallmatrix}1\\ \boldsymbol{\alpha}_1\end{smallmatrix},\cdots,\begin{smallmatrix}1\\ \boldsymbol{\alpha}_m\end{smallmatrix},\begin{smallmatrix}-1\\ -\boldsymbol{\beta}_1\end{smallmatrix},\cdots,\begin{smallmatrix}-1\\ -\boldsymbol{\beta}_l\end{smallmatrix}\right]$. Then,

$$
CA=\left[\sum_{j=1}^{l}\left(\sum_{i\in I_1}\begin{bmatrix}1\\ \boldsymbol{\alpha}_i\end{bmatrix}\lambda_{ij1}-\begin{bmatrix}1\\ \boldsymbol{\beta}_j\end{bmatrix}\right),\dots,\sum_{j=1}^{l}\left(\sum_{i\in I_r}\begin{bmatrix}1\\ \boldsymbol{\alpha}_i\end{bmatrix}\lambda_{ijr}-\begin{bmatrix}1\\ \boldsymbol{\beta}_j\end{bmatrix}\right)\right]=[\mathbf{0},\dots,\mathbf{0}],
$$

which implies that the row vectors of C belong to the cokernel of A. Since $\dim(\Delta_1)=n$, the volume of Δ_1, which equals $\frac{1}{n!}|\det(D)|$, where D is the matrix with column vectors $\left[\begin{smallmatrix}1\\ \boldsymbol{\alpha}_i\end{smallmatrix}\right]$, $i\in I_1$, is nonzero. It follows that $\mathrm{rank}(C)=n+1$. As $\dim(\mathrm{coker}(A))\leq n+1$, we then conclude that $\dim(\mathrm{coker}(A))=n+1$ and the row vectors of C span the cokernel of A. As a result, $\mathrm{rank}(A)=m+l-(n+1)>1$ and $\mathrm{rank}(A_{uv})=\mathrm{rank}(A)-1$. The zero $\mathbf{x}_*$ of f^* is also a minimizer of f^*. So it satisfies $\{f^*(\mathbf{x}_*)=0,\nabla f^*(\mathbf{x}_*)=\mathbf{0}\}$, which gives

$$
\begin{cases}
\sum_{i=1}^{m}c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}-\sum_{j=1}^{l-1}d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}-d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}=0,\\
\sum_{i=1}^{m}c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}-\sum_{j=1}^{l-1}d_j\boldsymbol{\beta}_j\mathbf{x}_*^{\boldsymbol{\beta}_j}-d_l^*\boldsymbol{\beta}_l\mathbf{x}_*^{\boldsymbol{\beta}_l}=\mathbf{0},
\end{cases}
\tag{4.6}
$$

i.e., $C \cdot [c_1\mathbf{x}_*^{\boldsymbol{\alpha}_1}, \ldots, c_m\mathbf{x}_*^{\boldsymbol{\alpha}_m}, d_1\mathbf{x}_*^{\boldsymbol{\beta}_1}, \ldots, d_{l-1}\mathbf{x}_*^{\boldsymbol{\beta}_{l-1}}, d_l^* x_*^{\boldsymbol{\beta}_l}]^\intercal = \mathbf{0}$. Thus by Lemma 3.6, (4.3) is consistent.

Now by Lemma 3.4, in order to prove the claim, we only need to show that every subsystem (4.4) in variables $\{s_{jk}\}_{j,k} \setminus \{s_{uv}\}$ has a nonnegative solution for all $u \in [l]$ and all $v \in [r]$. Given $u \in [l]$ and $v \in [r]$, from (4.5) we have $s_{up_i} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{iup_i}$ for $i \in [m] \setminus I_v$, $s_{jv} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}$ for $j \in [l-1] \setminus \{u\}$, and $s_{lv} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}$ if $u \neq l$. Hence

$$\begin{cases} s_{jk} = 0 & \text{for } j = u, k \in [r] \setminus (\{v\} \cup \{p_i \mid i \in [m] \setminus I_v\}) \text{ or } j \in [l] \setminus \{u\}, k \in [r] \setminus \{v\}, \\ s_{up_i} = \dfrac{c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{iup_i}} & \text{for } i \in [m] \setminus I_v, \\ s_{jv} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j} & \text{for } j \in [l-1] \setminus \{u\}, \\ s_{lv} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l} & \text{if } u \neq l \end{cases}$$

is a nonnegative solution to (4.4). So the claim is proved and the proof is completed. ■

Remark 4.2. When $\dim(\mathrm{New}(f)) = n$ and $m = n + 1$ (so $\mathrm{New}(f)$ is a simplex), the assumptions that some vertex of $\mathrm{New}(f)$ is simple and that all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ clearly hold. In this case, Theorem 4.1 identifies with Theorem 2.8. Therefore, Theorem 4.1 is a generalization of Theorem 2.8 to the case of polynomials with arbitrary Newton polytopes.

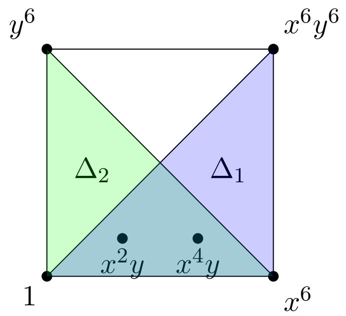
Remark 4.3. A polynomial of the form in Theorem 4.1 for which there exists a point $\boldsymbol{v} = [v_k] \in (\mathbb{R}^*)^n$ such that $d_j\boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j is called *orthant-dominated* in [9].

Example 4.4. Let $f_d = 1 + x^6 + y^6 + x^6y^6 - x^2y - dx^4y$ and $d^* = \sup\{d \in \mathbb{R}_{>0} \mid f_d \text{ is nonnegative}\}$. We have

$$\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}6\\6\end{bmatrix} + \frac{1}{6}\begin{bmatrix}6\\0\end{bmatrix} + \frac{2}{3}\begin{bmatrix}0\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}6\\0\end{bmatrix} + \frac{1}{6}\begin{bmatrix}0\\6\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}$$

and

$$\begin{bmatrix}4\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}6\\6\end{bmatrix} + \frac{1}{2}\begin{bmatrix}6\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0\\0\end{bmatrix} = \frac{2}{3}\begin{bmatrix}6\\0\end{bmatrix} + \frac{1}{6}\begin{bmatrix}0\\6\end{bmatrix} + \frac{1}{6}\begin{bmatrix}0\\0\end{bmatrix}.$$



The system of equations $\{f_d = 0, \nabla f_d = \mathbf{0}\}$ in variables $\{x, y, d\}$ has exactly one zero ($x_* \approx 1.04521, y_* \approx 0.764724, d^* \approx 2.11373$) in $\mathbb{R}^3_{>0}$. The linear system (4.3) becomes

$$(4.7)\qquad \begin{cases} \frac{2}{3}s_{11} + \frac{1}{2}s_{12} + \frac{1}{3}s_{21} + \frac{1}{6}s_{22} = 1, \\ \frac{1}{6}s_{11} + \frac{1}{3}s_{12} + \frac{1}{2}s_{21} + \frac{2}{3}s_{22} = x_*^6, \\ \frac{1}{6}s_{12} + \frac{1}{6}s_{22} = y_*^6, \\ \frac{1}{6}s_{11} + \frac{1}{6}s_{21} = x_*^6 y_*^6, \\ s_{11} + s_{12} = x_*^2 y_*, \\ s_{21} + s_{22} = d^* x_*^4 y_*, \end{cases}$$

which has a nonnegative solution ($s_{11} \approx 0.835429, s_{12} = 0, s_{21} \approx 0.729142, s_{22} = 1.2$). Thus, from the proof of Theorem 4.1, we obtain a SONC decomposition of f_{d^*}, which is

$$\begin{aligned} f_{d^*} \approx\ & (0.556953 + 0.106793x^6 + 0.533967x^6y^6 - x^2y) \\ & + (0.243047 + 0.27962x^6 + 0.466033x^6y^6 - 0.798909x^4y) \\ & + (0.2 + 0.613587x^6 + y^6 - 1.31482x^4y). \end{aligned}$$

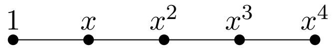
Corollary 4.5. *Let $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i \in \mathbb{R}_{>0}, i = 1, \dots, m$, $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ, d_j \in \mathbb{R}_{>0}, j = 1, \dots, l$, and $\dim(\mathrm{New}(f)) = n$. Assume that f is nonnegative and has a zero, that some vertex of $\mathrm{New}(f)$ is simple, and that all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}$. Then f has exactly one zero in $\mathbb{R}^n_{>0}$.*

Proof. By Theorem 4.1, f lies in the SONC cone. Suppose $f = \sum_{k=1}^r f_k$, where f_k is a nonnegative circuit polynomial for all $k \in [r]$. Let $\mathbf{x}_*$ be a zero of f. Then we have $f_k(\mathbf{x}_*) = 0$ for all $k \in [r]$. By Proposition 2.6, $f_k(|\mathbf{x}_*|) = 0$ and $|\mathbf{x}_*|$ is the only zero of f_k in $\mathbb{R}^n_{>0}$ for all k. Hence $|\mathbf{x}_*|$ is the only zero of f in $\mathbb{R}^n_{>0}$. ■

In the remainder of this section, we give an example to illustrate that the condition that all $\boldsymbol{\beta}_j$ lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}$ in Theorem 4.1 cannot be dropped.

Example 4.6. Let $f = 1 + 4x^2 + x^4 - 3x - 3x^3$. Then f is nonnegative, but f does not lie in the SONC cone.

Proof. As $f = (x-1)^2(x^2 - x + 1)$, its minimum is 0 and this is achieved at $x_* = 1$. By Theorem 5.6 (which will be proved in the next section), to get a SONC decomposition for f, it suffices to consider the circuits: $\{0, 2, 1\}, \{0, 4, 1\}, \{0, 4, 3\}, \{2, 4, 3\}$. We have $1 = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2 = \frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 4$, and $3 = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 4 = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 4$.



From the proof of Theorem 4.1, we have that if f lies in the SONC cone, then the linear system

$$
\begin{cases}
\frac{1}{2}s_1 + \frac{3}{4}s_3 + \frac{1}{4}s_4 = 1,\\
\frac{1}{2}s_1 + \frac{1}{2}s_3 = 4x_*^2,\\
\frac{1}{4}s_2 + \frac{1}{2}s_3 + \frac{3}{4}s_4 = x_*^4,\\
s_1 + s_2 = 3x_*,\\
s_3 + s_4 = 3x_*^3
\end{cases}
\tag{4.8}
$$

in variables $\{s_1, s_2, s_3, s_4\}$ should have a nonnegative solution. However, (4.8) has no nonnegative solution. This contradictory implies that f does not lie in the SONC cone. ■

5. SONC decompositions preserve sparsity. In this section, we prove the second main result of this paper: SONC decompositions preserve polynomial sparsity. To be more formal, for a nonnegative polynomial $f \in \mathbb{R}[\mathbf{x}]$, let $\Lambda(f) := \{\boldsymbol{\alpha} \in \operatorname{supp}(f) \mid \boldsymbol{\alpha} \in (2\mathbb{N})^n \text{ and } c_{\boldsymbol{\alpha}} > 0\}$ (corresponding to the positive terms) and $\Gamma(f) := \operatorname{supp}(f) \setminus \Lambda(f)$ (corresponding to the negative terms). Then we can write

$$
f = \sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \Gamma(f)} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}
$$

with $V(\operatorname{New}(f)) \subseteq \Lambda(f)$ (Proposition 2.1). For every $\boldsymbol{\beta} \in \Gamma(f)$, let

$$
\mathscr{F}(\boldsymbol{\beta}) := \{\Delta \mid \Delta \text{ is a simplex}, \boldsymbol{\beta} \in \Delta^{\circ}, V(\Delta) \subseteq \Lambda(f)\}. \tag{5.1}
$$

Consider the following SONC decomposition for f:

$$
f = \sum_{\boldsymbol{\beta} \in \Gamma(f)} \sum_{\Delta \in \mathscr{F}(\boldsymbol{\beta})} f_{\boldsymbol{\beta}\Delta} + \sum_{\boldsymbol{\alpha} \in \mathscr{I}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}, \tag{5.2}
$$

where $f_{\boldsymbol{\beta}\Delta}$ is a nonnegative circuit polynomial supported on $V(\Delta) \cup \{\boldsymbol{\beta}\}$ for all $\boldsymbol{\beta} \in \Gamma(f)$, $\Delta \in \mathscr{F}(\boldsymbol{\beta})$, and $\mathscr{I} = \{\boldsymbol{\alpha} \in \Lambda(f) \mid \boldsymbol{\alpha} \notin \bigcup_{\boldsymbol{\beta} \in \Gamma(f)} \bigcup_{\Delta \in \mathscr{F}(\boldsymbol{\beta})} V(\Delta)\}$. If f admits a SONC decomposition of the form (5.2), then we say that f decomposes into a *SONC polynomial without cancellation*.

In Theorems 3.9 and 4.1, we have seen that nonnegative polynomials satisfying certain conditions decompose into a SONC polynomial without cancellation. In this section, we shall prove that in fact every SONC decomposes into a sum of nonnegative circuit polynomials without cancellation. To this end, we first recall a connection between nonnegative circuit polynomials and SBS.

5.1. Nonnegative circuit polynomials and sums of binomial squares. For a subset $M \subseteq \mathbb{N}^n$, define $\overline{A}(M) := \{\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}) \mid \boldsymbol{u} \neq \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in M \cap (2\mathbb{N})^n\}$ as the set of averages of distinct even lattice points in M. For a trellis $\mathscr{A}$, we say that M is an *$\mathscr{A}$-mediated set* if $\mathscr{A} \subseteq M \subseteq \overline{A}(M) \cup \mathscr{A}$ [18]. It turns out that the problem whether a nonnegative circuit polynomial is an SOS is closely related to $\mathscr{A}$-mediated sets; see Theorem 5.2 in [5]. The following theorem states that for a nonnegative circuit polynomial $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}$, if $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set, then f is actually a SBS.

Theorem 5.1. *Let $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$, be a nonnegative circuit polynomial with $\boldsymbol{\beta} \in \operatorname{New}(f)^{\circ}$. If $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set M, then f is a SBS, i.e., $f = \sum_{2\boldsymbol{u}, 2\boldsymbol{v} \in M} (a_{\boldsymbol{u}} \mathbf{x}^{\boldsymbol{u}} - b_{\boldsymbol{v}} \mathbf{x}^{\boldsymbol{v}})^2$ for some $a_{\boldsymbol{u}}, b_{\boldsymbol{v}} \in \mathbb{R}$.*

Proof. The proof can be derived from Theorem 5.2 in [5] and Theorem 4.4 in [18]. ∎

Mediated sets were first studied by Reznick in [18]. For a trellis $\mathscr{A}$, there is a maximal $\mathscr{A}$-mediated set $\mathscr{A}^*$ satisfying $\overline{A}(\mathscr{A}) \subseteq \mathscr{A}^* \subseteq \operatorname{conv}(\mathscr{A}) \cap \mathbb{N}^n$ which contains every $\mathscr{A}$-mediated set. Following [18], we call a trellis $\mathscr{A}$ an *H-trellis* if $\mathscr{A}^* = \operatorname{conv}(\mathscr{A}) \cap \mathbb{N}^n$. The following theorem states that every trellis is an H-trellis after being multiplied by a sufficiently large integer.

Theorem 5.2 (see [16, Theorem 3.5]). *Let $\mathscr{A} \subseteq \mathbb{N}^n$ be a trellis. Then $k\mathscr{A}$ is an H-trellis for any integer $k \geq n$.*

Remark 5.3. The polynomials in [16] were assumed to be homogeneous. So we need $k \geq n$ instead of $k \geq n-1$ to adapt to our situation.

From Theorem 5.2 together with Theorem 5.1, we know that every n-variate nonnegative circuit polynomial supported on $k\mathscr{A}$ and a lattice point in the relative interior of $\operatorname{conv}(k\mathscr{A})$ is a SBS for any trellis $\mathscr{A}$ and an integer $k \geq n$.

Lemma 5.4. *Suppose that $f(x_1, \ldots, x_n) \in \mathbb{R}[\mathbf{x}]$ is a SONC. Then $f(x_1^k, \ldots, x_n^k)$ is a SBS for any integer $k \geq n$.*

Proof. Assume $f = \sum_i f_i$, where all f_i are nonnegative circuit polynomials. For any integer $k \geq n$, since every $f_i(x_1^k, \ldots, x_n^k)$ is a SBS (by Theorems 5.1 and 5.2), so is $f(x_1^k, \ldots, x_n^k)$. ∎

5.2. SONC decompostions without cancellation.

Now we prove that every SONC decomposes into a SONC polynomial without cancellation. The proof makes use of SBS decompositions for SONCs. The following lemma enables us to consider $f(x_1^k, \ldots, x_n^k)$ instead of $f(x_1, \ldots, x_n)$ for an odd number k.

Lemma 5.5. *Let $f(x_1, \ldots, x_n) \in \mathbb{R}[\mathbf{x}]$ and $k \in \mathbb{N}$ be an odd number. Then $f(x_1, \ldots, x_n)$ decomposes into a SONC polynomial without cancellation if and only if $f(x_1^k, \ldots, x_n^k)$ decomposes into a SONC polynomial without cancellation.*

Proof. Notice that for an odd number k, $f(x_1, \ldots, x_n)$ is a nonnegative circuit polynomial if and only if $f(x_1^k, \ldots, x_n^k)$ is a nonnegative circuit polynomial. The lemma then follows from this fact. ∎

If a nonnegative polynomial f can be written as

$$f = \sum_{\boldsymbol{\beta} \in \Gamma(f)} \left(\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \right) \tag{5.3}$$

such that every $\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$ is an AGE polynomial, then we say that f decomposes into a SAGE polynomials without cancellation. By Theorem 3.12, every AGE polynomial decomposes into a SONC polynomial without cancellation. Therefore, if f decomposes into a SAGE polynomial without cancellation, then f also decomposes into a SONC polynomial without cancellation.

Theorem 5.6. *Let* $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. *If* f *lies in the SONC cone, then* f *decomposes into a SONC polynomial without cancellation, i.e.,* f *admits a SONC decomposition of the form* (5.2).

Proof. By Lemma 5.5, we only need to prove the theorem for $f(x_1^{2n+1},\dots,x_n^{2n+1})$. In view of Theorem 3.12, we complete the proof by showing that $f(x_1^{2n+1},\dots,x_n^{2n+1})$ decomposes into a SAGE polynomial without cancellation.

For simplicity, let $h = f(x_1^{2n+1},\dots,x_n^{2n+1})$. By Lemma 5.4, we can write h as a SBS, i.e., $h = \sum_{i=1}^{m}(a_i\mathbf{x}^{\boldsymbol{u}_i} - b_i\mathbf{x}^{\boldsymbol{v}_i})^2$. To prove that h decomposes into a SAGE polynomial without cancellation, let us do induction on m. When $m=1$, $h = (a_1\mathbf{x}^{\boldsymbol{u}_1} - b_1\mathbf{x}^{\boldsymbol{v}_1})^2 = a_1^2\mathbf{x}^{2\boldsymbol{u}_1} + b_1^2\mathbf{x}^{2\boldsymbol{v}_1} - 2a_1b_1\mathbf{x}^{\boldsymbol{u}_1+\boldsymbol{v}_1}$ and the conclusion obviously holds. Assume that the conclusion is correct for $m-1$ and now consider the case of m. Let $h' = \sum_{i=1}^{m-1}(a_i\mathbf{x}^{\boldsymbol{u}_i} - b_i\mathbf{x}^{\boldsymbol{v}_i})^2 = \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(h')} d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. By the induction hypothesis, we can write $h' = \sum_{\boldsymbol{\beta}\in\Gamma(h')}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}})$ as a SAGE polynomial without cancellation. Then,

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h')}\left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}\right) + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2 . \tag{5.4}$$

From $h = h' + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2$, it follows that potential cancellation in (5.4) only occurs among terms involving $\mathbf{x}^{2\boldsymbol{u}_m}$, $\mathbf{x}^{2\boldsymbol{v}_m}$, $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$. Without loss of generality, we may assume $\boldsymbol{u}_m + \boldsymbol{v}_m \in \Gamma(h)$. Our goal is to rewrite h as a SAGE polynomial without cancellation by adjusting the terms involving $\mathbf{x}^{2\boldsymbol{u}_m}$, $\mathbf{x}^{2\boldsymbol{v}_m}$, $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ in (5.4).

First let us deal with the terms involving $\mathbf{x}^{2\boldsymbol{u}_m}$ by considering the following four cases.

Case I. If $2\boldsymbol{u}_m \notin \Gamma(h')$, then we have nothing to do.

Case II. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \in \Gamma(h)$, then we must have $d'_{2\boldsymbol{u}_m} > a_m^2$. By the equality

$$\begin{aligned}
&\left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}\right) + a_m^2\mathbf{x}^{2\boldsymbol{u}_m} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}\\
&\quad= \left(1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}}\right)\left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}\right)\\
&\qquad+ \sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m},
\end{aligned}$$

we obtain

$$\begin{aligned}
h = &\sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}}\left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}\right) + \left(1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}}\right)\left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}\right)\\
&+ \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}\right),
\end{aligned}$$

which is a SAGE polynomial with potential cancellation only occurring among terms involving $\mathbf{x}^{2\boldsymbol{v}_m}$, $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$.

Case III. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \in \Lambda(h)$, then we must have $a_m^2 > d'_{2\boldsymbol{u}_m}$, and we can write h as

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}} \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \right) + \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} + (a_m^2 - d'_{2\boldsymbol{u}_m}) \mathbf{x}^{2\boldsymbol{u}_m} + b_m^2 \mathbf{x}^{2\boldsymbol{v}_m} - 2a_m b_m \mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} \right),$$

which is a SAGE polynomial with potential cancellation only occurring among terms involving $\mathbf{x}^{2\boldsymbol{v}_m}$, $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$.

Case IV. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \notin \operatorname{supp}(h)$, then the terms $-d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}$ and $a_m^2\mathbf{x}^{2\boldsymbol{u}_m}$ must be cancelled in (5.4). Hence we obtain the expression of h as

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}} \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \right) + \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} + b_m^2 \mathbf{x}^{2\boldsymbol{v}_m} - 2a_m b_m \mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} \right),$$

which is a SAGE polynomial with potential cancellation only occurring among terms involving $\mathbf{x}^{2\boldsymbol{v}_m}$, $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$.

In much the same way, we continue to adjust the terms involving $\mathbf{x}^{2\boldsymbol{v}_m}$ in any of the above four cases, so that we can write h as

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h)\setminus\{\boldsymbol{u}_m+\boldsymbol{v}_m\}} \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c''_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d''_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \right) + \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h)} \tilde{c}_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \tilde{d}_{\boldsymbol{u}_m+\boldsymbol{v}_m} \mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} \right), \tag{5.5}$$

which is a SAGE polynomial with potential cancellation only occurring among terms involving $\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$. If $\boldsymbol{u}_m+\boldsymbol{v}_m \notin \Lambda(h')$, then (5.5) is already a SAGE polynomial without cancellation as desired. Assume now $\boldsymbol{u}_m+\boldsymbol{v}_m \in \Lambda(h')$. The assumption $\boldsymbol{u}_m+\boldsymbol{v}_m \in \Gamma(h)$ implies $\sum_{\boldsymbol{\beta}\in\Gamma(h)\setminus\{\boldsymbol{u}_m+\boldsymbol{v}_m\}} c''_{\boldsymbol{\beta}(\boldsymbol{u}_m+\boldsymbol{v}_m)} < \tilde{d}_{\boldsymbol{u}_m+\boldsymbol{v}_m}$ and so we can write (let $c''_{\boldsymbol{\beta}\boldsymbol{\alpha}} = 0$ for $\boldsymbol{\alpha} \in \Lambda(h)\setminus\Lambda(h')$)

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h)\setminus\{\boldsymbol{u}_m+\boldsymbol{v}_m\}} \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h)} \left(c''_{\boldsymbol{\beta}\boldsymbol{\alpha}} + \frac{\tilde{c}_{\boldsymbol{\alpha}} c''_{\boldsymbol{\beta}(\boldsymbol{u}_m+\boldsymbol{v}_m)}}{\tilde{d}_{\boldsymbol{u}_m+\boldsymbol{v}_m}} \right) \mathbf{x}^{\boldsymbol{\alpha}} - d''_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \right) + \left(1 - \frac{\sum_{\boldsymbol{\beta}\in\Gamma(h)\setminus\{\boldsymbol{u}_m+\boldsymbol{v}_m\}} c''_{\boldsymbol{\beta}(\boldsymbol{u}_m+\boldsymbol{v}_m)}}{\tilde{d}_{\boldsymbol{u}_m+\boldsymbol{v}_m}} \right) \left(\sum_{\boldsymbol{\alpha}\in\Lambda(h)} \tilde{c}_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \tilde{d}_{\boldsymbol{u}_m+\boldsymbol{v}_m} \mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} \right),$$

which is a SAGE polynomials without cancellation as desired. ■

Due to the coincidence of the SONC cone and the SAGE cone, it is immediate from Theorem 5.6 that every SAGE polynomial decomposes into a SAGE polynomials without cancellation. This result was also independently proved in [9, Corollary 20] by different techniques, where it was first proved in the context of signomials based on convex duality and then was specialized to the situation of polynomials. The proof given here, however, deals directly with polynomials and employs integrality of exponents in an essential way.

Theorem 5.6 ensures that every SONC admits a SONC decomposition by using only the support from the original polynomial and without cancellation. This is a very desired property (sparsity-preservation) to design efficient algorithms for sparse polynomial optimization based on SONC decompositions and is a distinguished difference from SOS decompositions. In the SOS case, a well-known result concerning sparsity due to Reznick states that if $f = \sum_i f_i^2$, then $\mathrm{supp}(f_i) \subseteq \frac{1}{2}\mathrm{New}(f)$ [17, Theorem 1], but generally cancellation occurs among f_i^2's. As a simple example, consider $f = 3 - 4x + x^4 = 2(1-x)^2 + (1-x^2)^2$. When we expand the squares on the right side, the monomial x^2 appears though it does not belong to the support of f.

When we apply SONC certificates to unconstrained polynomial optimization (i.e., minimizing a polynomial function over $\mathbb{R}^n$), the first problem is to decide which circuits are needed in construction of SONC decompositions. Once the set of candidate circuits is given, the rest of the computation can be done via relative entropy programming [5] or second order cone programming [24]. Hence the overall complexity largely depends on the number of candidate circuits. Thanks to Theorem 5.6, one may consider only the circuits that are contained in the support of the input polynomial instead of enumerating all possible circuits, so that the number of candidate circuits is enormously decreased. However, we emphasize that the number of circuits contained in a given set can be still large; see [9] for such an example in which the number of circuits is $2^{(m-1)/2}$ for a polynomial with m terms. In [7], the notion of reduced circuits was proposed so that one can remove slightly more redundant circuits. On the other hand, by Carathéodory's theorem [19, Corollary 17.1.2], it is possible to write a SONC f as a sum of at most $|\mathrm{supp}(f)|$ nonnegative circuit polynomials. We still do not know whether there are theoretical obstacles to stop us from obtaining such a SONC decomposition efficiently. In any case, more efforts are required to further reduce the number of candidate circuits and to make the computation more tractable.

As a comparison, when we apply SAGE certificates to unconstrained polynomial optimization, the number of AGE polynomials in construction of SAGE decompositions equals the number of negative terms of the input polynomial owing to Theorem 5.6, and deciding whether a polynomial is an AGE polynomial can be performed via relative entropy programming [9]. Thus the whole computation can be done efficiently for sparse polynomials.

6. Conclusions and discussions. This paper has studied several problems concerning SONC decompositions for nonnegative polynomials. We have proved that nonnegative polynomials with one negative term lie in the SONC cone. This result implies that the SONC cone actually coincides with the SAGE cone. We have also provided sufficient conditions for an arbitrary nonnegative polynomial to lie in the SONC cone. Moreover, we have proved that every SONC admits a SONC decomposition without cancellation. Following this line of research, there are still many questions left for further investigation:

- In Theorem 4.1, we have used a technical condition that some vertex of the Newton polytope is simple to complete the proof. It is not clear whether this condition can be dropped. The answer seems closely related to the existence of positive zeros for a particular system of polynomial equations [22].
- Even though the number of candidate circuits is significantly reduced thanks to the SONC decomposition without cancellation (5.2) we have provided, the computation of such a SONC decomposition is still generally intractable because the number of circuits used in (5.2) increases rapidly with the number of terms of the input polynomial. For unconstrained polynomial optimization, one may rely on certain heuristics to obtain a reasonable number of circuits as [20] or [24] did at the cost of losing some accuracy. One would also like to seek an approach to decrease the number of circuits without losing accuracy. The recent work in [14] made the first step toward this direction. See also the discussion at the end of section 5.2.
- The fact that every SONC after an appropriate dilation of the support is a SBS, which we rely on to prove Theorem 5.6, indicates the possibility of computing SONC decompositions via second order cone programming. The recent work in [24] is a good start on this topic.
- Another interesting and also important question is, To what extent can the results of this paper be generalized to the case of nonnegativity over a subset of $\mathbb{R}^n$? Does the sparsity-preserving property still hold for certain classes of subsets? The answers to these questions would help to use SONC certificates to solve constrained polynomial optimization problems. The recent work in [11, 12] can be viewed as attempts toward this direction.

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