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Toric Difference Variety*

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Abstract In this paper, the concept of toric difference varieties is defined and four equivalent descriptions for toric difference varieties are presented in terms of difference rational parametrization, difference coordinate rings, toric difference ideals, and group actions by difference tori. Connections between toric difference varieties and affine $\mathbb{N}[x]$-semimodules are established by proving the one-to-one correspondence between irreducible invariant difference subvarieties and faces of $\mathbb{N}[x]$-semimodules and the orbit-face correspondence. Finally, an algorithm is given to decide whether a binomial difference ideal represented by a $\mathbb{Z}[x]$-lattice defines a toric difference variety.

Keywords Affine $\mathbb{N}[x]$-semimodule, difference torus, T-orbit, toirc difference ideal, toric difference variety, $\mathbb{Z}[x]$-lattice.

1 Introduction

The theory of toric varieties has been extensively studied since its foundation in the early 1970s by Demazure[1], Miyake and Oda[2], Kempf, et al.[3], and Satake[4], due to its deep connections with polytopes, combinatorics, symplectic geometry, topology, and its applications in physics, coding theory, algebraic statistics, and hypergeometric functions[5–7].

In this paper, we initiate the study of toric difference varieties and expect that they will play similar roles in difference algebraic geometry to their algebraic counterparts in algebraic geometry. Difference algebra and difference algebraic geometry[8–11] were founded by Ritt and Doob[12] and Cohn[8], who aimed to study algebraic difference equations as algebraic geometry to polynomial equations.

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As in the algebraic case, a difference variety is said to be toric if it is the Cohn closure of the values of a set of Laurent difference monomials. To be more precise, we introduce the notion of symbolic exponents. For $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]$ and a in a difference ring over a difference field k with the difference operator σ, denote $a^p = \prod_{i=0}^{s}(\sigma^i(a))^{c_i}$. Then a Laurent difference monomial in the difference indeterminates $\mathbb{T} = (t_1, t_2, \cdots, t_n)$ has the form $\mathbb{T}^{\boldsymbol{u}} = \prod_{i=1}^{n} t_i^{u_i}$, where $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) \in \mathbb{Z}[x]^n$. For

$$U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}, \text{ where } \boldsymbol{u}_i \in \mathbb{Z}[x]^n, \quad i = 1, 2, \cdots, m, \tag{1}$$

define the following map

$$\theta_K : (K^*)^n \longrightarrow (K^*)^m, \mathbb{T} \mapsto \mathbb{T}^U = (\mathbb{T}^{\boldsymbol{u}_1}, \quad \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}), \tag{2}$$

where K is any difference extension field of k and $K^* = K \setminus \{0\}$. Then, the toric difference variety X_U defined by U is the Cohn closure of the image of θ.

A $\mathbb{Z}[x]$-lattice is a $\mathbb{Z}[x]$-submodule of $\mathbb{Z}[x]^n$, which plays the similar role as lattices do in the study of toric algebraic varieties. A $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^n$ is said to be toric if $g\boldsymbol{u} \in L$ implies $\boldsymbol{u} \in L$ for any $g \in \mathbb{Z}[x] \setminus \{0\}$ and $\boldsymbol{u} \in \mathbb{Z}[x]^n$. We will show that a difference variety $X \subseteq \mathbb{A}^m$ is toric if and only if the defining difference ideal for X is $\mathcal{I}_L = [\mathbb{Y}^{\boldsymbol{u}} - \mathbb{Y}^{\boldsymbol{v}} \mid \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}[x]^m \text{ with } \boldsymbol{u} - \boldsymbol{v} \in L]$, where L is a toric $\mathbb{Z}[x]$-lattice and $\mathbb{Y} = (y_1, y_2, \cdots, y_m)$ is a tuple of difference indeterminants. An algorithm is given to decide whether a $\mathbb{Z}[x]$-lattice is toric, and consequently, to decide whether $\mathcal{I}_L$ defines a toric difference variety.

As in the algebraic case, a difference variety X is toric if and only if X contains a difference torus T as a Cohn open subset and with a difference algebraic group action of T on X extending the natural group action of T on itself. Unlike the algebraic case, a difference torus is not necessarily isomorphic to $(\mathbb{A}^*)^m$, and this makes the definition of difference tori more complicated.

Many properties of toric difference varieties can be described using affine $\mathbb{N}[x]$-semimodules. An affine $\mathbb{N}[x]$-semimodule S generated by U in (1) is $\{\sum_{i=1}^{m} g_i \boldsymbol{u}_i \mid g_i \in \mathbb{N}[x]\}$. We will show that a difference variety X is toric if and only if $X \simeq \mathrm{Spec}^{\sigma}(k[S])$ for some affine $\mathbb{N}[x]$-semimodule S in $\mathbb{Z}[x]^n$, where $k[S] = \{\sum_{\boldsymbol{u} \in S} \alpha_{\boldsymbol{u}} \mathbb{T}^{\boldsymbol{u}} | \alpha_{\boldsymbol{u}} \in k, \alpha_{\boldsymbol{u}} \neq 0 \text{ for finitely many } \boldsymbol{u}\}$. Furthermore, there is a one-to-one correspondence between the irreducible invariant difference subvarieties of a toric difference variety and the faces of the corresponding affine $\mathbb{N}[x]$-semimodule. A one-to-one correspondence between the T-orbits of a toric difference variety and the faces of the corresponding affine $\mathbb{N}[x]$-semimodule is also established for a class of affine $\mathbb{N}[x]$-semimodules.

Toric difference varieties connect difference Chow forms[13] and sparse difference resultants[14]. Actually, the difference Chow form of X_U is the difference sparse resultant of the generic difference polynomials with monomials $\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}$. As a consequence, a Jacobi style order bound for a toric difference variety X_U is given.

The rest of this paper is organized as follows. In Section 2, preliminaries for difference algebra are introduced. In Section 3, the concept of difference toric variety is defined and its coordinate ring is given in terms of affine $\mathbb{N}[x]$-semimodules. In Section 4, the one-to-one correspondence between toric difference varieties and toric difference ideals is given. In Section 5, a description of toric difference varieties in terms of group actions is given. In Section 6, deeper connections between toric difference varieties and affine $\mathbb{N}[x]$-semimodules are given. In Section 7, an order bound for a toric difference variety is given. In Section 8, an algorithm is given to decide whether a given $\mathbb{Z}[x]$-lattice is $\mathbb{Z}[x]$-saturated. Conclusions are given in Section 9.

2 Preliminaries

We recall some basic notions from difference algebra. Standard references are [8, 10, 11]. All rings in this paper will be assumed to be commutative and unital.

A difference ring, or σ-ring for short, is a ring R together with a ring endomorphism $\sigma\colon R \to R$. If R is a field, then we call it a difference field, or a σ-field for short. A morphism between σ-rings R and S is a ring homomorphism $\psi\colon R \to S$ which commutes with the difference operators. In this paper, all σ-fields will be assumed to be of characteristic 0.

Let k be a σ-field. A k-algebra R is called a k-σ-algebra if the algebraic structure map $k \to R$ is a morphism of σ-rings. A morphism of k-σ-algebras is a morphism of k-algebras which is also a morphism of σ-rings. A k-subalgebra of a k-σ-algebra is called a k-σ-subalgebra if it is closed under σ. If a k-σ-algebra is a σ-field, then it is called a σ-field extension of k. Let R and S be two k-σ-algebras. Then $R \otimes_k S$ is naturally a k-σ-algebra by defining $\sigma(r \otimes s) = \sigma(r) \otimes \sigma(s)$ for $r \in R$ and $s \in S$.

Let k be a σ-field and R a k-σ-algebra. For a subset A of R, the smallest k-σ-subalgebra of R containing A is denoted by $k\{A\}$. If there exists a finite subset A of R such that $R = k\{A\}$, we say that R is finitely σ-generated over k. If moreover R is a σ-field, the smallest k-σ-subfield of R containing A is denoted by $k\langle A\rangle$.

Now we introduce the following useful notation. Let x be an algebraic indeterminate and $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]$. For a in a σ-ring R, denote $a^p = \prod_{i=0}^{s}(\sigma^i(a))^{c_i}$ with $\sigma^0(a) = a$ and $a^0 = 1$. It is easy to check that for $p, q \in \mathbb{Z}[x], a^{p+q} = a^p a^q, a^{pq} = (a^p)^q$.

Suppose $\mathbb{Y} = \{y_1, y_2, \cdots, y_m\}$ is a set of σ-indeterminates over k. Then the σ-polynomial ring over k in $\mathbb{Y}$ is the polynomial ring in the variables $\sigma^i(y_j)$ for $i \in \mathbb{N}$ and $j = 1, 2, \cdots, m$. It is denoted by $k\{\mathbb{Y}\} = k\{y_1, y_2, \cdots, y_m\}$ and has a natural k-σ-algebra structure. A σ-polynomial ideal, or simply a σ-ideal, I in $k\{\mathbb{Y}\}$ is an algebraic ideal which is closed under σ, i.e., $\sigma(I) \subseteq I$. If I also has the property that $\sigma(a) \in I$ implies $a \in I$, it is called a reflexive σ-ideal. A σ-prime ideal is a reflexive σ-ideal which is prime as an algebraic ideal. A σ-ideal I is called perfect if for any $g \in \mathbb{N}[x] \setminus \{0\}$ and $a \in k\{\mathbb{Y}\}$, $a^g \in I$ implies $a \in I$. It is easy to prove that every σ-prime ideal is perfect. If S is a set of σ-polynomials in $k\{\mathbb{Y}\}$, we use (S), $[S]$, and $\{S\}$ to denote the algebraic ideal, the σ-ideal, and the perfect σ-ideal generated by S respectively.

For $\boldsymbol{u} = (u_1, u_2, \cdots, u_m) \in \mathbb{Z}[x]^m$, $\mathbb{Y}^{\boldsymbol{u}} = \prod_{i=1}^{m} y_i^{u_i}$ is called a Laurent σ-monomial and $\boldsymbol{u}$ is called its support. A Laurent σ-polynomial in $\mathbb{Y}$ is a linear combination of Laurent σ-monomials and $k\{\mathbb{Y}^{\pm}\}$ denotes the set of all Laurent σ-polynomials, which is obviously a k-σ-algebra.

Let k be a σ-field. We denote the category of σ-field extensions of k by $\mathscr{E}_k$. Let $F \subseteq k\{\mathbb{Y}\}$

be a set of σ-polynomials. For any $K \in \mathscr{E}_k$, define the solutions of F in K to be

$$\mathbb{V}_K(F) := \{a \in K^m \mid f(a) = 0 \text{ for all } \ f \in F\}.$$

Note that $K \rightsquigarrow \mathbb{V}_K(F)$ is naturally a functor from the category of σ-field extensions of k to the category of sets. Denote this functor by $\mathbb{V}(F)$.

Definition 2.1 Let k be a σ-field. An (affine) difference variety or σ-variety over k is a functor X from the category of σ-field extensions of k to the category of sets which is of the form $\mathbb{V}(F)$ for some subset F of $k\{\mathbb{Y}\}$. In this situation, we say that X is the (affine) σ-variety defined by F.

Since in this paper we only consider the affine case, we will omit the word "affine" for short.

The functor $\mathbb{A}_k^m$ given by $\mathbb{A}_k^m(K) = K^m$ for $K \in \mathscr{E}_k$ is called the σ-affine (m-)space over k. If the base field k is specified, we often omit the subscript k.

By definition, a morphism of σ-varieties $\phi\colon X \to Y$ consists of maps $\phi_K\colon X(K) \to Y(K)$ for any $K \in \mathscr{E}_k$. If X and Y are two σ-varieties over k, then we write $X \subseteq Y$ to indicate that X is a subfunctor of Y. This simply means that $X(K) \subseteq Y(K)$ for every $K \in \mathscr{E}_k$. In this situation, we also say that X is a σ-subvariety of Y.

Let X be a σ-subvariety of $\mathbb{A}_k^m$. Then

$$\mathbb{I}(X) := \{f \in k\{\mathbb{Y}\} \mid f(a) = 0 \text{ for all } a \in X(K) \text{ and all } K \in \mathscr{E}_k\}$$

is called the vanishing ideal of X. It is well known that σ-subvarieties of $\mathbb{A}_k^m$ are in a one-to-one correspondence with perfect σ-ideals of $k\{\mathbb{Y}\}$ and we have $\mathbb{I}(\mathbb{V}(F)) = \{F\}$ for $F \subseteq k\{\mathbb{Y}\}$ (see reference [8, p.113]).

Definition 2.2 Let X be a σ-subvariety of $\mathbb{A}_k^m$. Then the k-σ-algebra

$$k\{X\} := k\{\mathbb{Y}\}/\mathbb{I}(X)$$

is called the σ-coordinate ring of X.

A k-σ-algebra is called an affine k-σ-algebra if it is isomorphic to $k\{\mathbb{Y}\}/\mathbb{I}(X)$ for some σ-variety X. Then by definition, $k\{X\}$ is an affine k-σ-algebra.

As in the case of affine algebraic varieties, the category of affine k-σ-varieties is antiequivalent to the category of affine k-σ-algebras (see [11, Theorem 2.1.21]). The following lemma is from [11, p.27].

Lemma 2.3 *Let X be a k-σ-variety. Then for any $K \in \mathscr{E}_k$, there is a natural bijection between $X(K)$ and the set of k-σ-algebra homomorphisms from $k\{X\}$ to K. Indeed,*

$$X \simeq \mathrm{Hom}(k\{X\}, -)$$

as functors.

Suppose that X is a k-σ-variety. Let $\mathrm{Spec}^\sigma(k\{X\})$ be the set of all σ-prime ideals of $k\{X\}$. Let $F \subseteq k\{X\}$. Set

$$\mathcal{V}(F) := \{\mathfrak{p} \in \mathrm{Spec}^\sigma(k\{X\}) \mid F \subseteq \mathfrak{p}\} \subseteq \mathrm{Spec}^\sigma(k\{X\}).$$

It can be checked that $\mathrm{Spec}^\sigma(k\{X\})$ is a topological space with closed sets of forms $\mathcal{V}(F)$. Then the topological space of X is $\mathrm{Spec}^\sigma(k\{X\})$ equipped with the above Cohn topology.

Let k be a σ-field and $F \subseteq k\{\mathbb{Y}\}$. Let $K, L \in \mathscr{E}_k$. Two solutions $a \in \mathbb{V}_K(F)$ and $b \in \mathbb{V}_L(F)$ are said to be equivalent if there exists a k-σ-isomorphism between $k\langle a\rangle$ and $k\langle b\rangle$ which maps a to b. Obviously, this defines an equivalence relation. The following theorem gives a relationship between equivalence classes of solutions of I and σ-prime ideals containing I ([11, Theorem 2.2.1]).

Theorem 2.4 *Let X be a k-σ-variety. There is a natural bijection between the set of equivalence classes of solutions of* $\mathbb{I}(X)$ *and* $\mathrm{Spec}^\sigma(k\{X\})$.

Because of Theorem 2.4, we shall not strictly distinguish between a σ-variety and its topological space. In other words, we use X to mean the σ-variety or its topological space.

3 Toric σ-Varieties

In this section, we will define toric σ-varieties and give a description of their coordinate rings in terms of affine $\mathbb{N}[x]$-semimodules.

Let k be a σ-field. Let $(\mathbb{A}^*)^n$ be the functor from $\mathscr{E}_k$ to $\mathscr{E}_k^n$ satisfying $(\mathbb{A}^*)^n(K) = (K^*)^n$, where $K \in \mathscr{E}_k$ and $K^* = K\backslash\{0\}$. In the rest of this section, we always assume

$$U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\} \subseteq \mathbb{Z}[x]^n \text{ and } \mathbb{T} = (t_1, t_2, \cdots, t_n), \tag{3}$$

a tuple of σ-indeterminates. We define the following map

$$\theta\colon (\mathbb{A}^*)^n \longrightarrow (\mathbb{A}^*)^m, \quad \mathbb{T} \mapsto \mathbb{T}^U = (\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}). \tag{4}$$

Define the functor T_U^* from $\mathscr{E}_k$ to $\mathscr{E}_k^m$ with $T_U^*(K) = \mathrm{Im}(\theta_K)$ for each $K \in \mathscr{E}_k$ which is called the quasi σ-torus defined by U.

Definition 3.1 A σ-variety over the σ-field k is said to be toric if it is the Cohn closure of a quasi σ-torus $T_U^* \subseteq \mathbb{A}^m$ in $\mathbb{A}^m$. Precisely, let

$$\mathcal{I}_U := \{f \in k\{\mathbb{Y}\} = k\{y_1, y_2, \cdots, y_m\} \mid f(\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}) = 0\}. \tag{5}$$

Then the (affine) toric σ-variety defined by U is $X_U = \mathbb{V}(\mathcal{I}_U)$.

Lemma 3.2 *X_U defined above is an irreducible σ-variety of σ-dimension* $\mathrm{rk}(\overline{U})$*, where* $\overline{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m]$ *is the matrix with $\boldsymbol{u}_i$ as the i-th column.*

Proof It is clear that $\mathbb{T}^U$ in (4) is a generic zero of $\mathcal{I}_U$ in (5). Then $\mathcal{I}_U$ is a σ-prime ideal. By Theorem 3.20 of [14], the σ-dimension of $\mathcal{I}_U$ is the difference transcendental degree of $k\langle\mathbb{T}^U\rangle$ over k, which is $\mathrm{rk}(\overline{U})$. ∎

Let $\mathbb{T}^{\pm} = \{t_1, t_2, \cdots, t_n, t_1^{-1}, t_2^{-1}, \cdots, t_n^{-1}\}$. Let $\mathcal{I}_{U,\mathbb{T}^{\pm}} = [y_1 - \mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, y_m - \mathbb{T}^{\boldsymbol{u}_m}]$ be the σ-ideal generated by $y_i - \mathbb{T}^{\boldsymbol{u}_i}, i = 1, 2, \cdots, m$ in $k\{\mathbb{Y}, \mathbb{T}^{\pm}\}$. Then it is easy to check

$$\mathcal{I}_U = \mathcal{I}_{U,\mathbb{T}^{\pm}} \cap k\{\mathbb{Y}\}. \tag{6}$$

Alternatively, let z be a new σ-indeterminate and $\mathcal{I}_{U,\mathbb{T}} = [\mathbb{T}^{\boldsymbol{u}_1^+}y_1 - \mathbb{T}^{\boldsymbol{u}_1^-}, \mathbb{T}^{\boldsymbol{u}_2^+}y_2 - \mathbb{T}^{\boldsymbol{u}_2^-}, \cdots, \mathbb{T}^{\boldsymbol{u}_m^+}y_m - \mathbb{T}^{\boldsymbol{u}_m^-}, \prod_{i=1}^n t_i z - 1]$ be a σ-ideal in $k\{\mathbb{Y}, \mathbb{T}, z\}$, where $\boldsymbol{u}_i^+, \boldsymbol{u}_i^- \in \mathbb{N}[x]^n$ are the postive and negative parts of $\boldsymbol{u}_i = \boldsymbol{u}_i^+ - \boldsymbol{u}_i^-$ respectively, $i = 1, 2, \cdots, m$. Then

$$\mathcal{I}_U = \mathcal{I}_{U,\mathbb{T}} \cap k\{\mathbb{Y}\}. \tag{7}$$

Equation (7) can be used to compute a characteristic set (see [15]) for $\mathcal{I}_U$ as shown in the following example.

Example 3.3 Let $M = \left[\begin{smallmatrix} 2 & x-1 & 0 & 0 \\ 0 & 0 & 2 & x-1 \end{smallmatrix}\right]$ and U the set of column vectors of M. Let $\mathcal{I}_1 = [y_1 - t_1^2, t_1y_2 - t_1^x, y_3 - t_2^2, t_2y_4 - t_2^x, t_1t_2z - 1]$. By (7), $\mathcal{I}_U = \mathcal{I}_1 \cap k\{y_1, y_2, y_3, y_4\}$. With the characteristic set method (see [15]), under the variable order $y_2 < y_4 < y_1 < y_3 < t_1 < t_2 < z$, a characteristic set of $\mathcal{I}_1$ is $y_1y_2^2 - y_1^x, y_3y_4^2 - y_3^x, y_1 - t_1^2, t_1y_2 - t_1^x, y_3 - t_2^2, t_2y_4 - t_2^x, t_1t_2z - 1$. Then $\mathcal{I}_U = \mathcal{I}_1 \cap k\{y_1, y_2, y_3, y_4\} = [y_1y_2^2 - y_1^x, y_3y_4^2 - y_3^x]$.

The following example shows that some y_i might not appear effectively in $\mathcal{I}_U$.

Example 3.4 Let $U = \{[1,1]^\tau, [x,x]^\tau, [0,1]^\tau\}$. By (7), $\mathcal{I}_U = [y_1 - t_1t_2, y_2 - t_1^xt_2^x, y_3 - t_2, t_1t_2z - 1] \cap k\{y_1, y_2, y_3\} = [y_1^x - y_2]$ and y_3 does not appear in $\mathcal{I}_U$.

Next, we will give a description for the coordinate ring of a toric σ-variety in terms of affine $\mathbb{N}[x]$-semimodules. A subset $S \subseteq \mathbb{Z}[x]^n$ is called an $\mathbb{N}[x]$-semimodule if it satisfies (i) for $\boldsymbol{a}, \boldsymbol{b} \in S$, $\mathbf{a} + \boldsymbol{b} \in S$; (ii) for $g \in \mathbb{N}[x]$ and $\boldsymbol{a} \in S$, $g\boldsymbol{a} \in S$. Moreover, if there exists a finite subset $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\} \subseteq \mathbb{Z}[x]^n$ such that $S = \mathbb{N}[x](U) = \{\sum_{i=1}^m g_i\boldsymbol{u}_i \mid g_i \in g_i\boldsymbol{u}_i \mid g_i \in \mathbb{N}[x]\}$, then S is called an affine $\mathbb{N}[x]$-semimodule. A map $\phi\colon S \to S'$ between two $\mathbb{N}[x]$-semimodules is an $\mathbb{N}[x]$-semimodule morphism if $\phi(\boldsymbol{a} + \boldsymbol{b}) = \phi(\boldsymbol{a}) + \phi(\boldsymbol{b})$ and $\phi(g\boldsymbol{a}) = g\phi(\boldsymbol{a})$ for all $\boldsymbol{a}, \boldsymbol{b} \in S, g \in \mathbb{N}[x]$.

Let k be a σ-field. For every affine $\mathbb{N}[x]$-semimodule S, we associate it with the following $\mathbb{N}[x]$-semimodule algebra $k[S]$ which is the vector space over k with S as a basis and has the multiplication induced by the addition of S. More concretely,

$$k[S] := \bigoplus_{\boldsymbol{u} \in S} k\mathbb{T}^{\boldsymbol{u}} = \left\{ \sum_{\boldsymbol{u} \in S} c_{\boldsymbol{u}}\mathbb{T}^{\boldsymbol{u}} \mid c_{\boldsymbol{u}} \in k \text{ and } c_{\boldsymbol{u}} = 0 \text{ for all but finitely many } \boldsymbol{u} \right\}$$

with the multiplication induced by $\mathbb{T}^{\boldsymbol{u}} \cdot \mathbb{T}^{\boldsymbol{v}} = \mathbb{T}^{\boldsymbol{u}+\boldsymbol{v}}$, for $\boldsymbol{u}, \boldsymbol{v} \in S$. Make $k[S]$ to be a k-σ-algebra by defining $\sigma(\mathbb{T}^{\boldsymbol{u}}) = \mathbb{T}^{x\boldsymbol{u}}$, for $\boldsymbol{u} \in S$.

If $S = \mathbb{N}[x](U) = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\})$, then $k[S] = k\{\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}\}$. Therefore, $k[S]$ is a finitely σ-generated k-σ-algebra. When an embedding $S \to \mathbb{Z}[x]^n$ is given, it induces an embedding $k[S] \to k[\mathbb{Z}[x]^n] \simeq k\{t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}\} = k\{\mathbb{T}^{\pm}\}$. So $k[S]$ is a k-σ-subalgebra of $k\{\mathbb{T}^{\pm}\}$ generated by finitely many Laurent σ-monomials and it follows that $k[S]$ is a σ-domain. We will see that $k[S]$ is actually the σ-coordinate ring of a toric σ-variety.

Theorem 3.5 *Let X be a σ-variety. Then X is a toric σ-variety if and only if there exists an affine $\mathbb{N}[x]$-semimodule S such that $X \simeq \mathrm{Spec}^\sigma(k[S])$. Equivalently, the σ-coordinate ring of X is $k[S]$.*

Proof Let $X = X_U$ be a toric σ-variety defined by U in (3) and $\mathcal{I}_U$ defined in (5). Let $S = \mathbb{N}[x](U)$ be the affine $\mathbb{N}[x]$-semimodule generated by U. Define the following morphism of

σ-rings

$$\theta\colon k\{\mathbb{Y}\} \longrightarrow k[S], \text{ where } \theta(y_i) = \mathbb{T}^{\boldsymbol{u}_i}, \quad i = 1, 2, \cdots, m.$$

The map θ is surjective by the definition of $k[S]$. If $f \in \ker(\theta)$, then $f(\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_i}) = 0$, which is equivalent to $f \in \mathcal{I}_U$. Then, $\ker(\theta) = \mathcal{I}_U$ and $k\{\mathbb{Y}\}/\mathcal{I}_U \simeq k[S]$. Therefore $X \simeq \mathrm{Spec}^\sigma(k\{\mathbb{Y}\}/\mathcal{I}_U) = \mathrm{Spec}^\sigma(k[S])$. Conversely, if $X \simeq \mathrm{Spec}^\sigma(k[S])$, where $S \subseteq \mathbb{Z}[x]^n$ is an affine $\mathbb{N}[x]$-semimodule, and $S = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\})$ for $\boldsymbol{u}_i \in S$. Let X_U be the toric σ-variety defined by $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$. Then as we just proved, the σ-coordinate ring of X is isomorphic to $k[S]$. Then $X \simeq X_U$. ∎

Suppose that S is an affine $\mathbb{N}[x]$-semimodule. For each $K \in \mathscr{E}_k$, a map $\varphi\colon S \to K$ is a morphism from S to K if φ satisfies $\varphi(\sum_i g_i \boldsymbol{u}_i) = \prod_i \varphi(\boldsymbol{u}_i)^{g_i}$, for $\boldsymbol{u}_i \in S$ and $g_i \in \mathbb{N}[x]$.

Proposition 3.6 *Let $S = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}) \subseteq \mathbb{Z}[x]^n$ be an affine $\mathbb{N}[x]$-semimodule and let $X = \mathrm{Spec}^\sigma(k[S])$ be the toric σ-variety associated with S. Then there is a one-to-one correspondence between $X(K)$ and $\mathrm{Hom}(S, K)$, for all $K \in \mathscr{E}_k$. Equivalently, $X \simeq \mathrm{Hom}(S, -)$ as functors.*

Proof Let $K \in \mathscr{E}_k$. By Lemma 2.3, an element of $X(K)$ is given by a k-σ-algebra homomorphism $f\colon k[S] \to K$, where $K \in \mathscr{E}_k$. Then f induces a morphism $\overline{f}\colon S \to K$ such that $\overline{f}(\boldsymbol{u}) = f(\mathbb{T}^{\boldsymbol{u}})$ for $\boldsymbol{u} \in S$. Conversely, given a morphism $\varphi\colon S \to K$, let $p = (\varphi(\boldsymbol{u}_1), \varphi(\boldsymbol{u}_2), \cdots, \varphi(\boldsymbol{u}_m)) \in K^m$. One can check that $p \in X(K)$. ∎

In the rest of this paper, we will identity elements of $X(K)$ with morphisms from S to K.

4 Toric σ-Ideal

In this section, we will show that toric σ-varieties are defined exactly by toric σ-ideals. We first define the concept of $\mathbb{Z}[x]$-lattice which is introduced in [16].

A $\mathbb{Z}[x]$-lattice is a $\mathbb{Z}[x]$-submodule of $\mathbb{Z}[x]^m$ for some m. Since $\mathbb{Z}[x]^m$ is Noetherian as a $\mathbb{Z}[x]$-module, we see that any $\mathbb{Z}[x]$-lattice is finitely generated. Let L be generated by $\mathbb{f} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_s\} \subseteq \mathbb{Z}[x]^m$, which is denoted as $L = (\mathbb{f})_{\mathbb{Z}[x]}$. Then the matrix with $\boldsymbol{f}_i$ as the i-th column is called a matrix representation of L. Define the rank of L, $\mathrm{rk}(L)$, to be the rank of its any representing matrix. Note that L may not be a free $\mathbb{Z}[x]$-module, thus the number of minimal generators of L can be larger than its rank.

A $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$ is said to be toric if it is $\mathbb{Z}[x]$-saturated, that is, for any nonzero $g \in \mathbb{Z}[x]$ and $\boldsymbol{u} \in \mathbb{Z}[x]^m$, $g\boldsymbol{u} \in L$ implies $\boldsymbol{u} \in L$.

Definition 4.1 Given a $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$, we define a binomial σ-ideal $\mathcal{I}_L \subseteq k\{\mathbb{Y}\} = k\{y_1, y_2, \cdots, y_m\}$

$$\mathcal{I}_L := [\mathbb{Y}^{\boldsymbol{u}^+} - \mathbb{Y}^{\boldsymbol{u}^-} \mid \boldsymbol{u} \in L] = [\mathbb{Y}^{\boldsymbol{u}} - \mathbb{Y}^{\boldsymbol{v}} \mid \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}[x]^m \text{with } \boldsymbol{u} - \boldsymbol{v} \in L],$$

where $\boldsymbol{u}^+, \boldsymbol{u}^- \in \mathbb{N}[x]^m$ are the positive part and the negative part of $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$, respectively. If L is toric, then the corresponding $\mathbb{Z}[x]$-lattice ideal $\mathcal{I}_L$ is called a toric σ-ideal.

$\mathcal{I}_L$ has the following properties.

1) Since a toric $\mathbb{Z}[x]$-lattice is both $\mathbb{Z}$-saturated and x-saturated, by Corollary 6.22 in [16], $\mathcal{I}_L$ is a σ-prime ideal of σ-dimension $m-\mathrm{rk}(L)$.

2) By Theorem 6.19 in [16], toric σ-ideals I_L in $k\{\mathbb{Y}\}$ are in a one-to-one correspondence with toric $\mathbb{Z}[x]$-lattices L in $\mathbb{Z}[x]^m$, that is, $L=\{\boldsymbol{u}-\boldsymbol{v}\,|\,\mathbb{Y}^{\boldsymbol{u}}-\mathbb{Y}^{\boldsymbol{v}}\in\mathcal{I}_L\}$. L is called the support lattice of $\mathcal{I}_L$.

In the rest of this section, we will prove the following result which can be deduced from Lemmas 4.3 and 4.5.

Theorem 4.2 *A σ-variety X is toric if and only if $\mathbb{I}(X)$ is a toric σ-ideal.*

Lemma 4.3 *Let X_U be the toric σ-variety defined in* (5)*. Then $\mathcal{I}_U=\mathbb{I}(X_U)$ is a toric σ-ideal whose support lattice is $L=\mathrm{Syz}(U)=\{\boldsymbol{f}\in\mathbb{Z}[x]^m\,|\,\overline{U}\boldsymbol{f}=\boldsymbol{0}\}$, where $\overline{U}=[\boldsymbol{u}_1,\boldsymbol{u}_2,\cdots,\boldsymbol{u}_m]$ is the matrix with columns $\boldsymbol{u}_i$.*

Proof L is clearly a toric $\mathbb{Z}[x]$-lattice. Then it suffices to show that $\mathcal{I}_U=\mathcal{I}_L$, where $\mathcal{I}_U$ is defined in (5). For $\boldsymbol{f}\in L$, we have $(\mathbb{Y}^{\boldsymbol{f}}-1)|_{\mathbb{Y}=\mathbb{T}^U}=(\mathbb{T}^U)^{\boldsymbol{f}}-1=\mathbb{T}^{\overline{U}\boldsymbol{f}}-1=0$. As a consequence, $(\mathbb{Y}^{\boldsymbol{f}^+}-\mathbb{Y}^{\boldsymbol{f}^-})|_{\mathbb{Y}=\mathbb{T}^U}=0$ and $\mathbb{Y}^{\boldsymbol{f}^+}-\mathbb{Y}^{\boldsymbol{f}^-}\in\mathcal{I}_U$. Since $\mathcal{I}_L$ is generated by $\mathbb{Y}^{\boldsymbol{f}^+}-\mathbb{Y}^{\boldsymbol{f}^-}$ for $\boldsymbol{f}\in L$, we have $\mathcal{I}_L\subseteq\mathcal{I}_U$.

To prove the other direction, consider a total order $\leq$ for σ-monomials $\{\mathbb{Y}^{\boldsymbol{f}},\boldsymbol{f}\in\mathbb{N}[x]^m\}$ (see reference [15]), which leads to a strict order $\prec$ on $\mathcal{F}\{\mathbb{Y}\}$: for $f,g\in\mathcal{F}\{\mathbb{Y}\}$, define $f\prec g$ if the largest σ-monomial of f is strictly less than that of g w.r.t $\leq$. We will prove $\mathcal{I}_U\subseteq\mathcal{I}_L$. Assume the contrary, and let $f=\Sigma a_i\mathbb{Y}^{\boldsymbol{f}_i}\in\mathcal{I}_{\mathbb{U}}$ be a minimal element in $\mathcal{I}_U\setminus\mathcal{I}_L$ under the above order. Let $a_0\mathbb{Y}^{\boldsymbol{g}}$ be the largest σ-monomial in f. From $f\in\mathcal{I}_U$, we have $f(\mathbb{T}^U)=0$. Since $\mathbb{Y}^{\boldsymbol{g}}|_{\mathbb{Y}=\mathbb{T}^U}=\mathbb{T}^{\overline{U}\boldsymbol{g}}$ is a σ-monomial on $\mathbb{T}$ and $f(\mathbb{T}^U)=0$, there exists another σ-monomial $b_0\mathbb{Y}^{\boldsymbol{h}}$ in f such that $\mathbb{Y}^{\boldsymbol{h}}|_{\mathbb{Y}=\mathbb{T}^U}=\mathbb{Y}^{\boldsymbol{g}}|_{\mathbb{Y}=\mathbb{T}^U}$. As a consequence, $(\mathbb{Y}^{\boldsymbol{g}}-\mathbb{Y}^{\boldsymbol{h}})|_{\mathbb{Y}=\mathbb{T}^U}=\mathbb{T}^{\overline{U}\boldsymbol{h}}(\mathbb{T}^{\overline{U}(\boldsymbol{g}-\boldsymbol{h})}-1)=0$, from which we deduce $\boldsymbol{g}-\boldsymbol{h}\in L$ and hence $\mathbb{Y}^{\boldsymbol{g}}-\mathbb{Y}^{\boldsymbol{h}}\in\mathcal{I}_U\cap\mathcal{I}_L$. Then $f-a_0(\mathbb{Y}^{\boldsymbol{g}}-\mathbb{Y}^{\boldsymbol{h}})\in\mathcal{I}_U\setminus\mathcal{I}_L$, which contradicts to the minimal property of f, since $f-a_0(\mathbb{Y}^{\boldsymbol{g}}-\mathbb{Y}^{\boldsymbol{h}})\prec f$. ∎

Let $L\subseteq\mathbb{Z}[x]^m$ be a $\mathbb{Z}[x]$-lattice. Define the orthogonal complement of L to be

$$L^C:=\{\boldsymbol{f}\in\mathbb{Z}[x]^m\mid\text{ for all }\boldsymbol{g}\in L,\langle\boldsymbol{f},\boldsymbol{g}\rangle=0\}$$

where $\langle\boldsymbol{f},\boldsymbol{g}\rangle=\boldsymbol{f}^\tau\cdot\boldsymbol{g}$ is the dot product of $\boldsymbol{f}$ and $\boldsymbol{g}$. It is easy to show that

Lemma 4.4 *Let $A_{m\times r}$ be a matrix representation of L. Then $L^C=\ker(A^\tau)=\{\boldsymbol{f}\in\mathbb{Z}[x]^m\mid A^\tau\boldsymbol{f}=\boldsymbol{0}\}$ and hence $\mathrm{rk}(L^C)=m-\mathrm{rk}(L)$. Furthermore, if L is a toric $\mathbb{Z}[x]$-lattice, then $L=(L^C)^C$.*

The following lemma shows that the inverse of Lemma 4.3 is also valid.

Lemma 4.5 *If $\mathcal{I}$ is a toric σ-ideal in $k\{\mathbb{Y}\}$, then $\mathbb{V}(\mathcal{I})$ is a toric σ-variety.*

Proof Since $\mathcal{I}$ is a toric σ-ideal, then the $\mathbb{Z}[x]$-lattice corresponding to $\mathcal{I}$, denoted by L, is toric. Suppose that $V=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\}\subseteq\mathbb{Z}[x]^m$ is a set of generators of L^C. Regard V as a matrix with columns $\boldsymbol{v}_i$ and let $U=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\cdots,\boldsymbol{u}_m\}\subseteq\mathbb{Z}[x]^n$ be the set of the row vectors of V. Consider the toric σ-variety X_U defined by U. To prove the lemma, it suffices to show $X_U=\mathbb{V}(\mathcal{I})$ or $\mathcal{I}_U=\mathcal{I}$. Since toric σ-ideals and toric $\mathbb{Z}[x]$-lattices are in a one-to-one correspondence, we only need to show $\mathrm{Syz}(U) = L$. This is clear since $\mathrm{Syz}(U) = \ker(V) = (L^C)^C = L$ by Lemma 4.5. ▮

Example 4.6 Use notations introduced in Example 3.3. Let $\boldsymbol{f}_1 = (1-x, 2, 0, 0)^\tau$, $\boldsymbol{f}_2 = (0, 0, 1-x, 2)^\tau$. Then $L = \ker(M) = (\boldsymbol{f}_1, \boldsymbol{f}_2)_{\mathbb{Z}[x]} \subseteq \mathbb{Z}[x]^4$. By Lemma 4.3, we have $\mathcal{I}_U = \mathcal{I}_L = [y_1y_2^2 - y_1^x, y_3y_4^2 - y_3^x]$. In Example 3.3, we need to use the difference characteristic set method to compute $\mathcal{I}_U$. Here, the only operation used to compute $\mathcal{I}_U$ is the Gröbner basis method for $\mathbb{Z}[x]$-lattices (see [17]).

Finally, we have the following effective version of Theorem 4.2.

Theorem 4.7 *A toric σ-variety X has the parametric representation X_U and the implicit representation $\mathcal{I}_L$, where U is given in* (3) *and $L = (\mathbb{f})_{\mathbb{Z}[x]}$ for $\mathbb{f} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_s\} \subseteq \mathbb{Z}[x]^m$. Then, there is a polynomial-time algorithm to compute U from $\mathbb{f}$ and vise versa.*

Proof The proofs of Lemmas 4.3 and 4.5 give the algorithms to compute $\mathbb{f}$ from U, and vice versa, provided we know how to compute a set of generators of $\ker(A)$ for a matrix A with entries in $\mathbb{Z}[x]$. In [17], a polynomial-time algorithm to compute Gröbner bases for $\mathbb{Z}[x]$-lattices is given. Combining this with Schreyer's Theorem on page 224 of [18], we have an algorithm to compute a Gröbner basis for $\ker(A)$ as a $\mathbb{Z}[x]$-module. Note that, when a Gröbner basis of the $\mathbb{Z}[x]$-lattice generated by the column vectors of A is given, the complexity to compute a Gröbner basis of $\ker(A)$ using Schreyer's Theorem is clearly polynomial. ▮

In other words, toric σ-varieties are unirational σ-varieties, and we have efficient implicitization and parametrization algorithms for them.

5 σ-Torus and Toric σ-Variety in Terms of Group Actions

In this section, we will define σ-tori and give another description of toric σ-varieties in terms of group actions by σ-tori.

Let T_U^* be the quasi σ-torus and X_U the toric σ-variety defined by $U \subseteq \mathbb{Z}[x]^n$ in (4). In the algebraic case, T_U^* is a variety. That is, $T_U^* = X_U \cap (\mathbb{C}^*)^m$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The following example shows that this is not valid in the difference case.

Example 5.1 In Example 3.3, $X_U = \mathbb{V}(\{y_1y_2^2 - y_1^x, y_3y_4^2 - y_3^x\})$. Let $P = (-1, 1, -1, -1) \in \mathbb{C}^4$. Then $P \in X_U(\mathbb{C})$. On the other hand, assume $P \in T_U^*(\mathbb{C})$ which means $((t_1)^2, (t_1)^{x-1}, (t_2)^2, (t_2)^{x-1}) = (-1, 1, -1, -1)$ or the σ-equations $t_1^2 + 1 = 0, t_1^x - t_1 = 0, t_2^2 + 1 = 0, t_2^x + t_2 = 0$ have a solution in $(\mathbb{C}^*)^2$. In what below, we will show that this is impossible. That is, $T_U^* \subsetneq X_U \cap (\mathbb{C}^*)^4$.

Let $\mathcal{I} = [t_1^2 + 1, t_1^x - t_1, t_2^2 + 1, t_2^x + t_2]$. We have $t_2^2 - t_1^2 = t_2^2 + 1 - (t_1^2 + 1) \in \mathcal{I}$. Then, $\mathbb{V}(\mathcal{I}) = \mathbb{V}(\mathcal{I} \cup \{t_2 - t_1\}) \cup \mathbb{V}(\mathcal{I} \cup \{t_2 + t_1\})$. Since $t_2^x + t_2 - (t_2 - t_1)^x - (t_2 - t_1) - (t_1^x - t_1) = 2t_1$. Then $\mathbb{V}(\mathcal{I} \cup \{t_2 - t_1\}) = \mathbb{V}(\mathcal{I} \cup \{t_2 - t_1, t_1\}) = \emptyset$. Similarly, $\mathbb{V}(\mathcal{I} \cup \{t_2 + t_1\}) = \emptyset$ and hence $\mathbb{V}(\mathcal{I}) = \emptyset$.

In order to define σ-tori, we need to introduce the concept of Cohn $*$-closure. $(\mathbb{A}^*)^m$ is isomorphic to the σ-variety defined by $\mathcal{I}_0 = [y_1z_1 - 1, y_2z_2 - 1, \cdots, y_mz_m - 1] \subseteq k\{\mathbb{Y}, \mathbb{Z}\}$ in

$(\mathbb{A})^{2m}$, where $\mathbb{Z} = (z_1, z_2, \cdots, z_m)$ is a tuple of σ-indeterminants. Furthermore, σ-varieties in $(\mathbb{A}^*)^m$ are in a one-to-one correspondence with σ-varieties contained in $\mathbb{V}(\mathcal{I}_0)$ via the map

$$\theta\colon (\mathbb{A}^*)^m \longrightarrow (\mathbb{A})^{2m} \tag{8}$$

defined by $\theta(a_1, a_2, \cdots, a_m) = (a_1, a_2, \cdots, a_m, a_1^{-1}, a_2^{-1}, \cdots, a_m^{-1})$. Let $V \subseteq (\mathbb{A}^*)^m$ and V_1 be the Cohn closure of $\theta(V)$ in $(\mathbb{A})^{2m}$. Then $\theta^{-1}(V_1)$ is called the Cohn $*$-closure of V.

Example 5.1 gives the motivation for the following definition.

Definition 5.2 A σ-torus is a σ-variety which is isomorphic to the Cohn $*$-closure of a quasi σ-torus in $(\mathbb{A}^*)^m$ for some m.

Lemma 5.3 *Let T_U^* be the quasi σ-torus defined by U, T_U the Cohn $*$-closure of T_U^*, and $\mathcal{I}_U$ defined in* (5). *Then T_U is isomorphic to* $\mathrm{Spec}^\sigma(k\{\mathbb{Y}, \mathbb{Z}\}/\widetilde{\mathcal{I}}_U)$ *where $\mathbb{Z} = (z_1, z_2, \cdots, z_m)$ is a tuple of σ-indeterminates and $\widetilde{\mathcal{I}}_U = [\mathcal{I}_U, y_1z_1 - 1, y_2z_2 - 1, \cdots, y_mz_m - 1]$ in $k\{\mathbb{Y}, \mathbb{Z}\}$. We say T_U is the σ-torus defined by U.*

Proof Let θ be defined in (8). Let $\widetilde{T}_U^* = \theta(T_U^*) \subseteq \mathbb{A}^{2m}$ and $\widetilde{T}_U$ the Cohn closure of $\widetilde{T}_U^*$ in $\mathbb{A}^{2m}$. Then $T_U = \theta^{-1}(\widetilde{T}_U)$ is the Cohn $*$-closure of T_U^* in $(\mathbb{A}^*)^n$. Since θ is clearly an isomorphism between $\widetilde{T}_U$ and T_U, it suffices to show that $\mathbb{I}(\widetilde{T}_U) = \widetilde{\mathcal{I}}_U$.

We have $\mathbb{I}(\widetilde{T}_U) = \{f \in k\{\mathbb{Y}, \mathbb{Z}\} \mid f(\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}, \mathbb{T}^{-\boldsymbol{u}_1}, \mathbb{T}^{-\boldsymbol{u}_2}, \cdots, \mathbb{T}^{-\boldsymbol{u}_m}) = 0\}$. It is clear that $\widetilde{\mathcal{I}}_U \subseteq \mathbb{I}(\widetilde{T}_U)$. If $f \in \mathbb{I}(\widetilde{T}_U)$, eliminate $z_1, z_2, \cdots, z_m$ from f by replacing z_i with $\frac{1}{y_i}$ and clear the denominates, we have $f_1 = \prod_{i=1}^m y_i^{t_i} f + f_0$, where $f_0 \in \mathcal{I}_0$. Substituting y_i by $\mathbb{T}^{\boldsymbol{u}_i}$ and z_i by $\mathbb{T}^{-\boldsymbol{u}_i}$, we have $f_1(\mathbb{T}^{\boldsymbol{u}_1}, \mathbb{T}^{\boldsymbol{u}_2}, \cdots, \mathbb{T}^{\boldsymbol{u}_m}) = 0$, and $f_1 \in \mathcal{I}_U$ follows. Then $\prod_{i=1}^m y_i^{t_i} f \in \widetilde{\mathcal{I}}_U$ and hence $\prod_{i=1}^m z_i^{t_i} y_i^{t_i} f = \prod_{i=1}^m (y_iz_i - 1 + 1)^{t_i} f = f + f_0 \in \widetilde{\mathcal{I}}_U$, where $f_0 \in \mathcal{I}_0$. Thus, $f \in \widetilde{\mathcal{I}}_U$. ▮

Corollary 5.4 *Let T_U and X_U be the σ-torus and the toric σ-variety defined by U, respectively. Then $T_U = X_U \cap (\mathbb{A}^*)^m$. As a consequence, T_U is a Cohn open subset of X_U.*

Proof From Lemma 5.3 and the fact $\mathcal{I}_U = \mathbb{I}(X_U)$, we have $T_U = X_U \cap (\mathbb{A}^*)^m$. ▮

Theorem 5.5 *Let T be a σ-variety. Then T is a σ-torus if and only if there exists a $\mathbb{Z}[x]$-lattice L such that $T \simeq \mathrm{Spec}^\sigma(k[L])$.*

Proof We follow the notations in Lemma 5.3. Suppose that T is defined by U and let $L = (U)_{\mathbb{Z}[x]}$. Since $T \simeq \widetilde{T_U}$, we just need to show the σ-coordinate ring of $\widetilde{T_U}$ is $k[L]$. By definition, $\widetilde{T_U}$ is the toric σ-variety defined by $U \cup (-U)$. Thus, the affine $\mathbb{N}[x]$-semimodule corresponding to $\widetilde{T_U}$ is $\mathbb{N}[x](U \cup (-U)) = L$ and hence the σ-coordinate ring of $\widetilde{T_U}$ is $k[L]$. Conversely, suppose U is a finite subset of $\mathbb{Z}[x]^n$ and $L = (U)_{\mathbb{Z}[x]}$. Then by the proof of the above necessity, U defines a σ-torus T_U whose σ-coordinate ring is $k[L]$. Since $T \simeq T_U$, T is a σ-torus. ▮

As a consequence, a σ-torus is also a toric σ-variety. An algebraic torus is isomorphic to $(\mathbb{C}^*)^m$ for some $m \in \mathbb{N}$ (see [5]). The following example shows that this is not valid in the difference case.

Example 5.6 Let $\boldsymbol{u}_1 = (2)$, $\boldsymbol{u}_2 = (x)$, and $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$. We claim that T_U is not isomorphic to $(\mathbb{A}^*)^1$. By Theorem 5.5, we need to show that $\mathcal{E}_1 = k\{t, t^{-1}\}$ is not isomorphic to $\mathcal{E}_2 = k\{s^2, s^{-2}, s^x, s^{-x}\}$, where t and s are σ-indeterminates. Suppose that the contrary, there is an isomorphism $\theta\colon \mathcal{E}_1 \to \mathcal{E}_2$ and $\theta(t) = p(s) \in \mathcal{E}_2$. Then there exists a $q(z) \in k\{z\}$ such that $s^2 = q(p(s))$ which is possible only if $q = z, p = s^2$. Since $s^x \in \mathcal{E}_2$, there exists an $r(z) \in k\{z\}$ such that $s^x = r(s^2)$ which is impossible.

Suppose that S is an affine $\mathbb{N}[x]$-semimodule. Let $(S)_{\mathbb{Z}[x]}$ be the $\mathbb{Z}[x]$-lattice generated by S. Let $X = \mathrm{Spec}^\sigma(k[S])$ and $T = \mathrm{Spec}^\sigma(k[(S)_{\mathbb{Z}[x]}])$. Following Proposition 3.6, let $\gamma\colon S \to K$ be an element of $X(K)$ which lies in $T(K)$. Since elements of $T(K)$ are invertible, $\gamma(S) \subseteq K^*$ and hence γ can be extended to $\widetilde{\gamma}\colon (S)_{\mathbb{Z}[x]} \to K^*$.

Proposition 5.7 *There is a one-to-one correspondence between* $T(K)$ *and* $\mathrm{Hom}((S)_{\mathbb{Z}[x]}, K^*)$, *for all* $K \in \mathscr{E}_k$. *Equivalently,* $T \simeq \mathrm{Hom}((S)_{\mathbb{Z}[x]}, -)$ *as functors.* So we can identity an element of $T(K)$ with a morphism from $(S)_{\mathbb{Z}[x]}$ to K^*.

A σ-variety G is called a σ-algebraic group if G has a group structure and the group multiplication and the inverse map are both morphisms of σ-varieties (see [19]).

Lemma 5.8 *A* σ*-torus* T *is a* σ*-algebraic group.*

Proof For each $K \in \mathscr{E}_k$ and $\varphi, \psi \in T(K)$, define $\varphi \cdot \psi = \varphi\psi$. It is easy to check that $T(K)$ becomes a group under the multiplication. Note that if $T \subseteq (\mathbb{A}^*)^m$, the group multiplication of T is just the usual termwise multiplication of $\mathbb{A}^m$, namely, for $(x_1, x_2, \cdots, x_m), (y_1, y_2, \cdots, y_m) \in T$, $(x_1, x_2, \cdots, x_m) \cdot (y_1, y_2, \cdots, y_m) = (x_1y_1, x_2y_2, \cdots, x_my_m)$. So it is obviously a morphism of σ-varieties and so is the inverse map due to (8). Therefore, T is a σ-algebraic group. ∎

We now interpret what is a σ-algebraic group action on a σ-variety.

Definition 5.9 Let G be a σ-algebraic group and X a σ-variety. We say G has a σ-algebraic group action on X if there exists a morphism of σ-varieties

$$\phi\colon G \times X \longrightarrow X$$

such that for any $K \in \mathscr{E}_k$,

$$\phi_K\colon G(K) \times X(K) \longrightarrow X(K)$$

is a group action of $G(K)$ on $X(K)$.

The following theorem gives a description of toric σ-varieties in terms of group actions.

Theorem 5.10 *A* σ*-variety* X *is toric if and only if* X *contains a* σ*-torus* T *as an open subset and with a* σ*-algebraic group action of* T *on* X *extending the natural* σ*-algebraic group action of* T *on itself.*

Proof " $\Rightarrow$ ". By Corollary 5.4, T_U is an open subset of X_U. By Lemma 5.8, T_U is a σ-algebraic group. To show that T_U acts on X_U as a σ-algebraic group, define a map $\phi\colon X \times X \to X, \phi((x_1, x_2, \cdots, x_m), (y_1, y_2, \cdots, y_m)) = (x_1y_1, x_2y_2, \cdots, x_my_m)$. It can be described using $\mathbb{N}[x]$-semimodule morphisms as follows: For each $K \in \mathscr{E}_k$, let $\varphi, \psi\colon S \to K$ be two elements of $X(K)$, then $\phi((\varphi, \psi)) = \varphi\psi$. This corresponds to the k-σ-algebra homomorphism $\Phi\colon k[S] \to k[S] \otimes_k k[S]$ such that $\Phi(\mathbb{T}^{\boldsymbol{u}}) = \mathbb{T}^{\boldsymbol{u}} \otimes \mathbb{T}^{\boldsymbol{u}}$, for $\boldsymbol{u} \in S$. Via the embedding $T \subseteq X$, the operation on X induces a map $\phi\colon T \times X \to X$ which is clearly a σ-algebraic group action on X and extends the group action of T on itself.

" $\Leftarrow$ ". There is a $\mathbb{Z}[x]$-lattice L such that $T \simeq \mathrm{Spec}^\sigma(k[L])$. The open immersion $T \subseteq X$ induces $k\{X\} \subseteq k[L]$. Since the action of T on itself extends to a σ-algebraic group action on

X, we have the following commutative diagram:

$$\begin{tikzcd} T \times T \arrow[r, "\phi"] \arrow[d] & T \arrow[d] \\ T \times X \arrow[r, "\widetilde{\phi}"] & X \end{tikzcd} \tag{9}$$

where ϕ is the group action of T, $\widetilde{\phi}$ is the extension of ϕ to $T \times X$. From (9), we obtain the following commutative diagram of the corresponding σ-coordinate rings:

$$\begin{tikzcd} k\{X\} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & k[L] \otimes_k k\{X\} \arrow[d] \\ k[L] \arrow[r, "\Phi"] & k[L] \otimes_k k[L] \end{tikzcd}$$

where the vertical maps are inclusions, and $\Phi(\mathbb{T}^{\boldsymbol{u}}) = \mathbb{T}^{\boldsymbol{u}} \otimes \mathbb{T}^{\boldsymbol{u}}$ for $\boldsymbol{u} \in L$. It follows that if $\sum_{\boldsymbol{u}\in L} \alpha_{\boldsymbol{u}}\mathbb{T}^{\boldsymbol{u}}$ with finitely many $\alpha_{\boldsymbol{u}} \neq 0$ is in $k\{X\}$, then $\sum_{\boldsymbol{u}\in L} \alpha_{\boldsymbol{u}}\mathbb{T}^{\boldsymbol{u}} \otimes \mathbb{T}^{\boldsymbol{u}}$ is in $k[L] \otimes_k k\{X\}$, so $\alpha_{\boldsymbol{u}}\mathbb{T}^{\boldsymbol{u}} \in k\{X\}$ for every $\boldsymbol{u} \in L$. This shows that there is a subset S of L such that $k\{X\} = k[S] = \bigoplus_{\boldsymbol{u}\in S} k\mathbb{T}^{\boldsymbol{u}}$. Since $k\{X\}$ is a k-σ-subalgebra of $k[L]$, it follows that S is an $\mathbb{N}[x]$-semimodule. And $\mathbb{N}[x]$-semimodule. And since $k\{X\}$ is a finitely σ-generated k-σ-algebra, S is finitely generated, thus it is an affine $\mathbb{N}[x]$-semimodule. So by Theorem 3.5, X is a toric σ-variety. ∎

6 Toric σ-Varieties and Affine $\mathbb{N}[x]$-Semimodules

In this section, deeper connections between toric σ-varieties and affine $\mathbb{N}[x]$-semimodules will be established. We first show that the category of toric σ-varieties with toric morphisms is antiequivalent to the category of affine $\mathbb{N}[x]$-semimodules with $\mathbb{N}[x]$-semimodule morphisms.

Note that if $\phi\colon S_1 \to S_2$ is a morphism between two affine $\mathbb{N}[x]$-semimodules, we have an induced k-σ-algebra homomorphism $\overline{\phi}\colon k[S_1] \to k[S_2]$ such that $\overline{\phi}(\mathbb{T}^{\boldsymbol{u}}) = \mathbb{T}^{\phi(\boldsymbol{u})}, \boldsymbol{u} \in S$, which gives a morphism between toric σ-varieties $\phi^*\colon X_2 = \mathrm{Spec}^\sigma(k[S_2]) \to X_1 = \mathrm{Spec}^\sigma(k[S_1])$.

Definition 6.1 Let $X_i = \mathrm{Spec}^\sigma(k[S_i])$ be the toric σ-varieties coming from affine $\mathbb{N}[x]$-semimodules $S_i, i = 1, 2$ with σ-tori T_i, respectively. A morphism $\phi\colon X_1 \to X_2$ is said to be toric if $\phi(T_1) \subseteq T_2$ and $\phi|_{T_1}$ is a σ-algebraic group homomorphism.

Proposition 6.2 *Let $\phi\colon X_1 \to X_2$ be a toric morphism of toric σ-varieties. Then ϕ preserves group actions, namely, $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$ for all $t \in T_1$ and $p \in X_1$.*

Proof Suppose the action of T_i on X_i is given by a morphism $\varphi_i\colon T_i \times X_i \to X_i, i = 1, 2$. Preserving group action means that the following diagram is commutative:

$$\begin{tikzcd} T_1 \times X_1 \arrow[r, "\varphi_1"] \arrow[d, "\phi|_{T_1}\times\phi"'] & X_1 \arrow[d, "\phi"] \\ T_2 \times X_2 \arrow[r, "\varphi_2"] & X_2 \end{tikzcd} \tag{10}$$

If we replace X_i by T_i in the diagram, then it certainly commutes since $\phi|_{T_1}$ is a group homomorphism. Since $T_1 \times T_1$ is dense in $T_1 \times X_1$, the whole diagram is commutative. ∎

Lemma 6.3 *Let $T_i = \mathrm{Spec}^{\sigma}(k[L_i])$ be two σ-tori defined by the $\mathbb{Z}[x]$-lattices $L_i, i = 1, 2$. Then a map $\phi\colon T_1 \to T_2$ is a σ-algebraic group homomorphism if and only if the corresponding map of σ-coordinate rings $\phi^*\colon k[L_2] \to k[L_1]$ is induced by a $\mathbb{Z}[x]$-module homomorphism $\widehat{\phi}\colon L_2 \to L_1$.*

Proof "$\Leftarrow$". Suppose that $\widehat{\phi}\colon L_2 \to L_1$ is a $\mathbb{Z}[x]$-module homomorphism and it induces a morphism of σ-varieties $\phi\colon T_1 \to T_2$ via ϕ^*. Then, for any $\varphi, \psi \in T_1$, $\phi(\varphi \cdot \psi) = (\varphi \cdot \psi) \circ \widehat{\phi} = (\varphi \circ \widehat{\phi}) \cdot (\psi \circ \widehat{\phi}) = \phi(\varphi) \cdot \phi(\psi)$. So ϕ preserves group multiplications and hence is a morphism of σ-algebraic groups.

"$\Rightarrow$". Suppose that $\phi\colon T_1 \to T_2$ is a morphism of σ-algebraic groups. Then we have the following commutative diagram:

$$\begin{array}{ccc} k[L_2] & \longrightarrow & k[L_2] \otimes_k k[L_2] \\ \Big\downarrow{\scriptstyle \phi^*} & & \Big\downarrow{\scriptstyle \phi^* \otimes \phi^*} \\ k[L_1] & \longrightarrow & k[L_1] \otimes_k k[L_1] \end{array}$$

Given $\boldsymbol{v} \in L_2$, there is a finite subset S of L_1 such that $\phi^*(\mathbb{T}^{\boldsymbol{v}}) = \sum_{\boldsymbol{u} \in S} \alpha_{\boldsymbol{u}} \mathbb{T}^{\boldsymbol{u}}$. It follows from the commutativity of the diagram that $\sum_{\boldsymbol{u} \in S} \alpha_{\boldsymbol{u}} \mathbb{T}^{\boldsymbol{u}} \otimes \mathbb{T}^{\boldsymbol{u}} = \sum_{\boldsymbol{u}_1 \in S, \boldsymbol{u}_2 \in a} \alpha_{\boldsymbol{u}_1} \alpha_{\boldsymbol{u}_2} \mathbb{T}^{\boldsymbol{u}_1} \otimes \mathbb{T}^{\boldsymbol{u}_2}$. This shows that there is at most one $\boldsymbol{u}$ with $\alpha_{\boldsymbol{u}} \neq 0$ and in this case $\alpha_{\boldsymbol{u}} = 1$. Note that $\mathbb{T}^{\boldsymbol{v}}$ is invertible in $k[L_2]$, so $\phi^*(\mathbb{T}^{\boldsymbol{v}}) \neq 0$. So we have $\phi^*(\mathbb{T}^{\boldsymbol{v}}) = \mathbb{T}^{\boldsymbol{u}}$ for some $\boldsymbol{u} \in L_1$. Then we can define a map $\widehat{\phi}\colon L_2 \to L_1$, $\boldsymbol{v} \mapsto \boldsymbol{u}$. It is easy to check that $\widehat{\phi}$ is a $\mathbb{Z}[x]$-module homomorphism. ∎

Lemma 6.4 *Let $X_i = \mathrm{Spec}^{\sigma}(k[S_i])$ be the toric σ-varieties coming from affine $\mathbb{N}[x]$-semimodules $S_i, i = 1, 2, 2$ with σ-tori T_i respectively. Then a morphism $\phi\colon X_1 \to X_2$ is toric if and only if it is induced by an $\mathbb{N}[x]$-semimodule morphism $\widehat{\phi}\colon S_2 \to S_1$.*

Proof "$\Leftarrow$". Suppose that $\widehat{\phi}\colon S_2 \to S_1$ is an $\mathbb{N}[x]$-semimodule morphism. Then $\widehat{\phi}$ extends to a $\mathbb{Z}[x]$-module homomorphism $\widehat{\phi}\colon L_2 \to L_1$, where $L_1 = (S_1)_{\mathbb{Z}[x]}, L_2 = (S_2)_{\mathbb{Z}[x]}$. By Lemma 6.3, it induces a morphism of σ-algebraic groups $\phi\colon T_1 \to T_2$. So ϕ is toric.

"$\Rightarrow$". Since ϕ is toric, $\phi|_{T_1}$ is a σ-algebraic group homomorphism. By Lemma 6.3, it is induced by a $\mathbb{Z}[x]$-module homomorphism $\widehat{\phi}\colon L_2 \to L_1$. This, combined with $\phi^*(k[S_2]) \subseteq k[S_1]$, implies that $\widehat{\phi}$ induces an $\mathbb{N}[x]$-semimodule morphism $\widehat{\phi}\colon S_2 \to S_1$. ∎

Combining Theorem 3.5 with Lemma 6.4, we have the following theorem.

Theorem 6.5 *The category of toric σ-varieties with toric morphisms is antiequivalent to the category of affine $\mathbb{N}[x]$-semimodules with $\mathbb{N}[x]$-semimodule morphisms.*

In the rest of this section, we establish a one-to-one correspondence between irreducible T-invariant σ-subvarieties of a toric σ-variety and faces of the corresponding affine $\mathbb{N}[x]$-semimodule. Also, a one-to-one correspondence between T-orbits and faces of affine $\mathbb{N}[x]$-semimodules is given for a class of affine $\mathbb{N}[x]$-semimodules.

Definition 6.6 Let S be an affine $\mathbb{N}[x]$-semimodule. Define a face of S to be an $\mathbb{N}[x]$-subsemimodule $F \subseteq S$ such that

1) for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in S$, $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in F$ implies $\boldsymbol{u}_1, \boldsymbol{u}_2 \in F$;

2) for $\boldsymbol{u} \in S$, $x\boldsymbol{u} \in F$ implies $\boldsymbol{u} \in F$,

which is denoted by $F \preceq S$.

Note that if $S = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\})$ and F is a face of S, then F is generated by a subset of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$ as an $\mathbb{N}[x]$-semimodule. It follows that F is an affine $\mathbb{N}[x]$-semimodule and S has only finitely many faces. S is a face of itself. It is easy to prove that the intersection of two faces is again a face and a face of a face is again a face. S is said to be pointed if $S \cap (-S) = \{\mathbf{0}\}$, i.e., $\{\mathbf{0}\}$ is a face of S.

Example 6.7 Let $S = \mathbb{N}[x](\{\boldsymbol{u}_1 = (x, 1), \boldsymbol{u}_2 = (x, 2), \boldsymbol{u}_3 = (x, 3)\})$. Then S has four faces: $F_1 = \{0\}$, $F_2 = \mathbb{N}[x](\{\boldsymbol{u}_1\})$, $F_3 = \mathbb{N}[x](\{\boldsymbol{u}_3\})$ and $F_4 = S$. Since $2\boldsymbol{u}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, $\boldsymbol{u}_2$ does not generate a face.

For an $\mathbb{N}[x]$-semimodule $S \subseteq \mathbb{Z}[x]^n$, a σ-monomial in $k[S]$ is an element of the form $\mathbb{T}^{\boldsymbol{u}}$ with $\boldsymbol{u} \in S$. If we define a degree map by $\deg(\mathbb{T}^{\boldsymbol{u}}) = \boldsymbol{u}$, for $\boldsymbol{u} \in S$, then $k[S]$ becomes an S-graded ring. A σ-ideal of $k[S]$ is called S-homogeneous if it can be generated by homogeneous elements, i.e., σ-monomials.

Lemma 6.8 *A subset of F of S is a face if and only if $k[S\backslash F]$ is a σ-prime ideal of $k[S]$.*

Proof Let $I = k[S\backslash F]$. Since I is S-homogeneous, we just need to consider homogeneous elements, that is σ-monomials (see [20, Propsition 3.6]). The conditions for I to be a σ-ideal are that 1) if $\boldsymbol{u}_1 \in S\backslash F$ or $\boldsymbol{u}_2 \in S\backslash F$, then $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in S\backslash F$, and 2) if $\boldsymbol{u} \in S\backslash F$, then $x\boldsymbol{u} \in S\backslash F$, i.e., 1) $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in F$ implies $\boldsymbol{u}_1, \boldsymbol{u}_2 \in F$, and 2) $x\boldsymbol{u} \in F$ implies $\boldsymbol{u} \in F$. The condition for I to be prime is that if $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in S\backslash F$, then $\boldsymbol{u}_1 \in S\backslash F$ or $\boldsymbol{u}_2 \in S\backslash F$, i.e., $\boldsymbol{u}_1, \boldsymbol{u}_2 \in F$ implies $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in F$. The condition for I to be reflexive is that if $x\boldsymbol{u} \in S\backslash F$, then $\boldsymbol{u} \in S\backslash F$, i.e., $\boldsymbol{u} \in F$ implies $x\boldsymbol{u} \in F$. Putting all above together, we have that F is a face of S if and only if I is a σ-prime ideal. ∎

Let $X = \mathrm{Spec}^{\sigma}(k[S])$ be a toric σ-variety and T the σ-torus of X. A σ-subvariety Y of X is said to be invariant under the action of T if $T \cdot Y \subseteq Y$. For a face F of S, let $Y = \mathrm{Spec}^{\sigma}(k[F])$. Without loss of generality, assume that $S = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\})$ and $F = \mathbb{N}[x](\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r\})$. We always view Y as a σ-subvariety of X through the embedding $j\colon Y \to X, \gamma \in Y(K) \mapsto (\gamma(\boldsymbol{u}_1), \gamma(\boldsymbol{u}_2), \cdots, \gamma(\boldsymbol{u}_r), 0, 0, \cdots, 0) \in X(K)$ for each $K \in \mathscr{E}_k$. The following theorem gives a description for irreducible invariant σ-subvarieties of X.

Theorem 6.9 *Let $X = \mathrm{Spec}^{\sigma}(k[S])$ be a toric σ-variety and T the σ-torus of X. The irreducible invariant σ-subvarieties of X under the action of T are in an inclusion-preserving bijection with the faces of S. More precisely, if we denote the irreducible invariant σ-subvariety corresponding to the face F by $D(F)$, then $D(F)$ is defined by the σ-ideal $k[S\backslash F] = \bigoplus_{\boldsymbol{u} \in S\backslash F} k\mathbb{T}^{\boldsymbol{u}}$ and the σ-coordinate ring of $D(F)$ is $k[F] = \bigoplus_{\boldsymbol{u} \in F} k\mathbb{T}^{\boldsymbol{u}}$.*

Proof For a face F of S, let $Y = \mathrm{Spec}^{\sigma}(k[F])$. It is clear that Y is invariant under the action of T. The defining ideal of Y is $I = k[S\backslash F]$. Hence by Lemma 6.8, Y is irreducible.

On the other hand, let $L = (S)_{\mathbb{Z}[x]}$. Suppose that Y is an invariant irreducible σ-subvariety of X and is defined by the σ-ideal I. Then $k\{Y\} = k[S]/I$. By definition, Y is invariant under the σ-torus action if and only if the action of T on X induces an action on Y, that is, we have the following commutative diagram:

$$\begin{tikzcd} k[S] \arrow[r, "\phi"] \arrow[d] & k[L]\otimes k[S] \arrow[d] \\ k\{Y\} \arrow[r] & k[L]\otimes k\{Y\} \end{tikzcd}$$

Since $k[L]\otimes k\{Y\} = k[L]\otimes (k[S]/I) \simeq k[L]\otimes k[S]/k[L]\otimes I$, we must have $\phi(I)\subseteq k[L]\otimes I$. As in the proof of Theorem 5.10, this is equivalent with the fact that I is an L-graded ideal of $k[S]$. That is to say, we can write $I = \oplus_{\boldsymbol{u}\in S'} k\mathbb{T}^{\boldsymbol{u}}$, where S' is a subset of S. Since I is a σ-prime ideal, by Lemma 6.8, $F = S\backslash S'$ is a face of S. Moreover, since $I = k[S\backslash F]$, $k\{Y\} = k[S]/I = k[F]$. ∎

Note that an element $\gamma\colon S\to K$ of $X(K)$ lies in $D(F)(K)$ if and only if $\gamma(S\backslash F) = 0$ for any $K\in\mathscr{E}_k$.

Suppose that X is a toric σ-variety with σ-torus T. By Theorem 5.10, for each $K\in\mathscr{E}_k$, $T(K)$ has a group action on $X(K)$, so we have orbits of $T(K)$ in $X(K)$ under the action. To construct a correspondence between orbits and faces, we need a new kind of affine $\mathbb{N}[x]$-semimodules. An affine $\mathbb{N}[x]$-semimodule S is said to be face-saturated if for any face F of S, a morphism $\gamma\colon F\to K^*$ can be extended to a morphism $\widetilde{\gamma}\colon S\to K^*$ for any $K\in\mathscr{E}_k$. A necessary condition for S to be face-saturated is that for any face F of S, $(F)_{\mathbb{Z}[x]}$ is $\mathbb{N}[x]$-saturated in $(S)_{\mathbb{Z}[x]}$, that is, for $g\in\mathbb{N}[x]\backslash\{0\}$ and $\boldsymbol{u}\in(S)_{\mathbb{Z}[x]}$, $g\boldsymbol{u}\in(F)_{\mathbb{Z}[x]}$ implies $\boldsymbol{u}\in(F)_{\mathbb{Z}[x]}$.

Example 6.10 Let $S = \mathbb{N}[x](\{(2,0),(1,1),(0,1)\})$ and $F = \mathbb{N}[x](\{(2,0)\})$ a face of S. We have $(1,0)\in(S)_{\mathbb{Z}[x]}$. Since $(1,0)\notin F$ and $2(1,0)\in F$, S is not face-saturated.

Now we prove the following Orbit-Face correspondence theorem.

Theorem 6.11 *Suppose that S is a face-saturated affine $\mathbb{N}[x]$-semimodule. Let $X = \mathrm{Spec}^\sigma(k[S])$ be the toric σ-variety of S and T the σ-torus of X. Then for each $K\in\mathscr{E}_k$, there is a one-to-one correspondence between the faces of S and the orbits of $T(K)$ in $X(K)$.*

Proof For a face F of S, let $Y = \mathrm{Spec}^\sigma(k[F])$. The inclusion $F\subseteq S$ induces a morphism of toric σ-varieties $f\colon X\to Y$ and a morphism of σ-tori $g\colon T\to T_Y$, where $T_Y = \mathrm{Spec}^\sigma(k[(F)_{\mathbb{Z}[x]}])$. For each $K\in\mathscr{E}_k$, an element of $T_Y(K)$ is a morphism $\gamma\colon S\to K$ such that $\gamma(F)\subseteq K^*$ and $\gamma(S\backslash F) = 0$. Since S is face-saturated, γ can be extended to a morphism $\widetilde{\gamma}\colon S\to K^*$ which is an element of $T(K)$. So $g_K\colon T(K)\to T_Y(K)$ is surjective. Suppose that $\psi\colon S\to K^*$ is another element of $T(K)$, then the action of ψ on γ is $\psi\gamma$ which is still an element of $T_Y(K)$. So $T_Y(K)$ is closed under the action of $T(K)$. Suppose that e is the identity element of $T_Y(K)$, then $T(K)\cdot e = g_K(T(K))\cdot e$. Since g_K is surjective, $g_K(T(K)) = T_Y(K)$ and $T(K)\cdot e = T_Y(K)$. Therefore, $T_Y(K)$ is transitive under the action of $T(K)$. Thus, $T_Y(K)$ is an orbit for the action of $T(K)$ on $X(K)$.

On the other hand, for each $K\in\mathscr{E}_k$, given a morphism $\varphi\colon S\to K$ in $X(K)$, let $F := \varphi^{-1}(K^*)$. Then for $\boldsymbol{u}_1,\boldsymbol{u}_2\in F$ and $g_1,g_2\in\mathbb{N}[x]$, since $\varphi(\boldsymbol{u}_1g_1+\boldsymbol{u}_2g_2) = \varphi(\boldsymbol{u}_1)^{g_1}\varphi(\boldsymbol{u}_2)^{g_2}\in K^*$, $\boldsymbol{u}_1g_1+\boldsymbol{u}_2g_2\in F$. Therefore, F is an $\mathbb{N}[x]$-subsemimodule of S. Moreover, for $\boldsymbol{u}_1,\boldsymbol{u}_2\in S$, if $\boldsymbol{u}_1+\boldsymbol{u}_2\in F$, then $\varphi(\boldsymbol{u}_1+\boldsymbol{u}_2) = \varphi(\boldsymbol{u}_1)\varphi(\boldsymbol{u}_2)\in K^*$, from which it follows $\varphi(\boldsymbol{u}_1),\varphi(\boldsymbol{u}_2)\in K^*$ and hence $\boldsymbol{u}_1,\boldsymbol{u}_2\in F$. For $\boldsymbol{u}\in S$, if $x\boldsymbol{u}\in F$, then $\varphi(x\boldsymbol{u}) = \varphi(\boldsymbol{u})^x\in K^*$, from which it follows

$\varphi(\boldsymbol{u}) \in K^*$ and hence $\boldsymbol{u} \in F$. So F is a face of S. Let $Y = \mathrm{Spec}^\sigma(k[F]), T_Y = \mathrm{Spec}^\sigma(k[(F)_{\mathbb{Z}[x]}])$. It is clear that $\varphi \in T_Y(K)$ and $T_Y(K)$ is the orbit of φ in $X(K)$.

It is clear that two different faces give two discrete orbits, which proves the one-to-one correspondence. ▮

7 An Order Bound of Toric σ-Variety

In this section, we show that the σ-Chow form (see [13, 21]) of a toric σ-variety X_U is the sparse σ-resultant (see [14]) with support U. As a consequence, we can give a bound for the order of X_U.

Let $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\}$ be a subset of $\mathbb{Z}[x]^n$ and X_U the toric σ-variety defined by U. In order to establish a connection between the σ-Chow form of X_U and the sparse σ-resultant with support U, we assume that U is Laurent transformally essential (see [14]), that is $\mathrm{rk}(\overline{U}) = n$ by regrading $\overline{U}$ as a matrix with $\boldsymbol{u}_i$ as the i-th column.

Let $\mathbb{T} = \{t_1, t_2, \cdots, t_n\}$ be a set of σ-indeterminates. Here, the fact that U is Laurent transformally essential means that there exist indices $k_1, k_2, \cdots, k_n \in \{1, 2, \cdots, m\}$ such that the Laurent σ-monomials $\mathbb{T}^{\boldsymbol{u}_{k_1}}, \mathbb{T}^{\boldsymbol{u}_{k_2}}, \cdots, \mathbb{T}^{\boldsymbol{u}_{k_n}}$ are transformally independent over k (see [14]).

Let $\mathcal{A} = \{M_1 = \mathbb{T}^{\boldsymbol{u}_1}, M_2 = \mathbb{T}^{\boldsymbol{u}_2}, \cdots, M_m = \mathbb{Y}^{\boldsymbol{u}_m}\}$ and

$$\mathbb{P}_i = a_{i0} + a_{i1}M_1 + \cdots + a_{im}M_m, \quad i = 0, 1, \cdots, n. \tag{11}$$

$n+1$ generic Laurent σ-polynomials with the same support U. Denote $\boldsymbol{a}_i = (a_{i0}, a_{i1}, \cdots, a_{im})$, $i = 0, 1, \cdots, n$. Since $\mathcal{A}$ is Laurent transformally essential, the sparse σ-resultant of $\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n$ exists (see [14]), which is denoted by $R_U \in k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$.

By Lemma 3.2, $X_U \subseteq \mathbb{A}^m$ is an irreducible σ-variety of dimension $\mathrm{rk}(U) = n$. Then, the σ-Chow form of X_U, denoted by $C_U \in k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$, can be obtained by intersecting X_U with the following generic σ-hyperplanes (see [13])

$$\mathbb{L}_i = a_{i0} + a_{i1}y_1 + \cdots + a_{im}y_m, \quad i = 0, 1, \cdots, n.$$

We have

Theorem 7.1 *Up to a sign, the sparse σ-resultant R_U of $\mathbb{P}_i\,(i = 0, 1, \cdots, n)$ is the same as the σ-Chow form C_U of X_U.*

Proof All σ-ideals in this proof are supposed to be in $R = k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \mathbb{Y}, \mathbb{T}^{\pm}\}$, unless specifically mentioned otherwise. From [14],

$$[\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = \mathrm{sat}(R_U, R_1, R_2, \cdots, R_l)$$

is a σ-prime ideal of codimension one in $k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$. Let $\mathcal{I}_U = \mathbb{I}(X_U)$. From [13],

$$[\mathcal{I}_U, \mathbb{L}_0, \mathbb{L}_1, \cdots, \mathbb{L}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = \mathrm{sat}(C_U, C_1, C_2, \cdots, C_t)$$

is a σ-prime ideal of codimension one in $k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$. By Theorem 7 of [14], in order to prove $C_U = R_U$, it suffices to show

$$[\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots \boldsymbol{a}_n\} = [\mathcal{I}_U, \mathbb{L}_0, , \mathbb{L}_1, \cdots, \mathbb{L}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots \boldsymbol{a}_n\}.$$

Let $\mathcal{I}_{\mathbb{T}} = [y_1-M_1, y_2-M_2, \cdots, y_m-M_m]$. By (6), $\mathcal{I}_U = \mathcal{I}_{\mathbb{T}} \cap k\{\mathbb{Y}\}$. Then, $[\mathcal{I}_U, \mathbb{L}_0, \mathbb{L}_1, \cdots, \mathbb{L}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = [y_1-M_1, y_2-M_2, \cdots, y_m-M_m, \mathbb{L}_0, \mathbb{L}_1, \cdots, \mathbb{L}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = [y_1-M_1, y_2-M_2, \cdots, y_m-M_m, \mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$. Since $\mathbb{P}_i \in k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_m, \mathbb{T}^{\pm}\}$ does not contain any $y_i^{x^j}$, we have $[y_1-M_1, y_2-M_2, \cdots, y_m-M_m, \mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\} = [\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n] \cap k\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$, and the theorem is proved. ∎

To give a bound for the order of X_U, we need to introduce the concept of Jacobi number. Let $M = (m_{ij})$ be an $n \times n$ matrix with elements either in $\mathbb{N}$ or $-\infty$. A diagonal sum of M is any sum $m_{1\tau(1)} + m_{2\tau(2)} + \cdots + m_{n\tau(n)}$ with τ a permutation of $1, 2, \cdots, n$. The Jacobi number of M is the maximal diagonal sum of M, denoted by $\mathrm{Jac}(M)$ (see [14]).

Let $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\} \subseteq \mathbb{Z}[x]^n$ and $\overline{U} = (a_{ij})_{m \times n}$ the matrix with $\boldsymbol{u}_i$ as the i-th column. For each $i \in \{1, 2, \cdots, n\}$, let $o_i = \max_{k=1}^n \deg(a_{ik}, x)$ and assume that $\deg(0, x) = -\infty$. Since $\overline{U}$ does not contain zero rows, no a_{ij} is $-\infty$. For a $p(x) \in \mathbb{Z}[x]$, let $\underline{\deg}(p, x) = \min\{k \in \mathbb{N} \,|\, \text{s.t. } \mathrm{coeff}(p, x^k) \neq 0\}$ and $\underline{\deg}(0, x) = 0$. For each $i \in \{1, 2, \cdots, n\}$, let $\underline{o}_i = \min_{k=1}^n \underline{\deg}(a_{ik}, x)$ and $\underline{o} = \sum_{i=1}^n \underline{o}_i$.

Theorem 7.2 *Use the notations just introduced. Let X_U be the toric σ-variety defined by U. Then* $\mathrm{ord}(X_U) \leq \sum_{i=1}^n (o_i - \underline{o}_i)$.

Proof Use the notations in Theorem 7.1. Since $\mathbb{P}_i$ in (11) have the same support for all i, $\mathrm{ord}(R_U, \boldsymbol{a}_i)$ are the same for all i. The order matrix for $\mathbb{P}_i$ given in (11) is $O = (\mathrm{ord}(\mathbb{P}_i, t_j))_{(n+1)\times n} = (o_{ij})_{(n+1)\times n}$, where $o_{ij} = o_j$. That is, all rows of O are the same. Let $\overline{O}$ be obtained from O by deleting any row of O. Then $J = \mathrm{Jac}(\overline{O}) = \sum_{i=1}^n o_i$. By Theorem 4.17 of [14], $\mathrm{ord}(R_U, \boldsymbol{a}_i) \leq J - \underline{o} = \sum_{i=1}^n (o_i - \underline{o}_i)$. By Theorem 6.12 of [13], $\mathrm{ord}(X_U) = \mathrm{ord}(C_U, \boldsymbol{a}_i)$ for each $i = 0, 1, \cdots, n$. By Theorem 7.1, $C_U = R_U$. Then the theorem is proved. ∎

8 Algorithms

In this section, we give algorithms to decide whether a given $\mathbb{Z}[x]$-lattice L is toric and in the negative case to compute the $\mathbb{Z}[x]$-saturation of L. Using these algorithms, for a $\mathbb{Z}[x]$-lattice L generated by $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\} \subseteq \mathbb{Z}[x]^n$, we can decide whether the binomial σ-ideal $\mathcal{I}_L$ is toric and in the negative case, to compute a toric $\mathbb{Z}[x]$-lattice $L' \supseteq L$ such that $\mathcal{I}_{L'} \supseteq \mathcal{I}_L$ is the smallest toric σ-ideal containing $\mathcal{I}_L$.

We first introduce the concept of Gröbner bases for $\mathbb{Z}[x]$-lattices. For the details, please refer to [16, 18]. Denote $\boldsymbol{\varepsilon}_i$ to be the i-th standard basis vector $(0, 0, \cdots, 0, 1, 0, \cdots, 0)^\tau \in \mathbb{Z}[x]^n$, where 1 lies in the i-th row of $\boldsymbol{\varepsilon}_i$. A monomial $\boldsymbol{m}$ in $\mathbb{Z}[x]^n$ is an element of the form $ax^k\boldsymbol{\varepsilon}_i \in \mathbb{Z}[x]^n$, where $a \in \mathbb{Z}$ and $k \in \mathbb{N}$. The following monomial order $<$ of $\mathbb{Z}[x]^n$ will be used in this paper: $ax^\alpha\boldsymbol{\varepsilon}_i < bx^\beta\boldsymbol{\varepsilon}_j$ if $i < j$, or $i = j$ and $\alpha < \beta$, or $i = j$, $\alpha = \beta$, and $|a| < |b|$.

For any $\boldsymbol{f} \in \mathbb{Z}[x]^n$, the largest monomial in $\boldsymbol{f}$ is called the leading term of $\boldsymbol{f}$, which is denoted by $\mathbf{LT}(\boldsymbol{f})$. The order $<$ can be extended to elements of $\mathbb{Z}[x]^n$ as follows: For $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{Z}[x]^n$, $\boldsymbol{f} < \boldsymbol{g}$ if and only if $\mathbf{LT}(\boldsymbol{f}) < \mathbf{LT}(\boldsymbol{g})$.

A monomial $ax^\alpha\boldsymbol{\varepsilon}_i$ is said to be reduced w.r.t another nonzero monomial $bx^\beta\boldsymbol{\varepsilon}_j$ if one of the following three conditions holds:

1) $i \neq j$,

2) $i=j, \alpha<\beta$,

3) $i=j, \alpha\ge\beta$, and $0\le a<|b|$.

Let $\mathbb{G}\subseteq\mathbb{Z}[x]^n$ and $\boldsymbol{f}\in\mathbb{Z}[x]^n$. We say that $\boldsymbol{f}$ is reduced with respect to $\mathbb{G}$ if any monomial of $\boldsymbol{f}$ is not a multiple of $\mathbf{LT}(\boldsymbol{g})$ by an element in $\mathbb{Z}[x]$ for any $\boldsymbol{g}\in\mathbb{G}$.

A finite set $\mathbb{f}=\{\boldsymbol{f}_1,\boldsymbol{f}_2,\cdots,\boldsymbol{f}_s\}\subseteq\mathbb{Z}[x]^n$ is called a Gröbner basis for the $\mathbb{Z}[x]$-lattice L generated by $\mathbb{f}$ if for any $\boldsymbol{g}\in L$, there exists an i, such that $\mathbf{LT}(\boldsymbol{g})|\mathbf{LT}(\boldsymbol{f}_i)$. A Gröbner basis $\mathbb{f}$ is called reduced if for any $\boldsymbol{f}\in\mathbb{f}$, $\boldsymbol{f}$ is reduced with respect to $\mathbb{f}\setminus\{\boldsymbol{f}\}$.

Let $\mathbb{f}$ be a Gröbner basis. Then any $\boldsymbol{f}\in\mathbb{Z}[x]^n$ can be reduced to a unique normal form by $\mathbb{f}$, denoted by $\mathbf{grem}(\boldsymbol{f},\mathbb{f})$, which is reduced with respect to $\mathbb{f}$.

Let $\boldsymbol{f},\boldsymbol{g}\in\mathbb{Z}[x]^n$, $\mathbf{LT}(\boldsymbol{f})=ax^k\boldsymbol{e}_i$, $\mathbf{LT}(\boldsymbol{g})=bx^s\boldsymbol{e}_j$, $s\le k$. The S-polynomial of $\boldsymbol{f}$ and $\boldsymbol{g}$ is defined as follows: if $i\ne j$ then $S(\boldsymbol{f},\boldsymbol{g})=\mathbf{0}$; otherwise

$$S(\boldsymbol{f},\boldsymbol{g})=\begin{cases}\boldsymbol{f}-\dfrac{a}{b}x^{k-s}\boldsymbol{g}, & \text{if } b\mid a;\\[2mm] \dfrac{b}{a}\boldsymbol{f}-x^{k-s}\boldsymbol{g}, & \text{if } a\mid b;\\[2mm] u\boldsymbol{f}+vx^{k-s}\boldsymbol{g}, & \text{if } a\nmid b \text{ and } b\nmid a,\ \text{where } gcd(a,b)=ua+vb.\end{cases} \tag{12}$$

Then, it is known that $\mathbb{f}\subseteq\mathbb{Z}[x]^n$ is a Gröbner basis if and only if $\mathbf{grem}(S(\boldsymbol{f}_i,\boldsymbol{f}_j),\mathbb{f})=\mathbf{0}$ for all i,j (see [17, 18]).

Next, we will give the structure for the matrix representation for the Gröbner basis of a $\mathbb{Z}[x]$-lattice. Let

$$\mathcal{C}=\begin{bmatrix}
c_{1,1} & \cdots & c_{1,l_1} & c_{1,l_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
c_{r_1,1} & \cdots & c_{r_1,l_1} & c_{r_1,l_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & 0 & c_{r_1+1,1} & \cdots & c_{r_1+1,l_2} & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & 0 & c_{r_2,1} & \cdots & c_{r_2,l_2} & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & c_{r_{t-1}+1,1} & \cdots & c_{r_{t-1}+1,l_t}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & c_{r_t,1} & \cdots & c_{r_t,l_t}
\end{bmatrix}_{n\times s}, \tag{13}$$

whose elements are in $\mathbb{Z}[x]$. We denote by $\boldsymbol{c}_i$ to be the i-th column of $\mathcal{C}$ and by $\boldsymbol{c}_{i,j}$ to be the column whose r_i-th element is $c_{r_i,j}$ for $i=1,2,\cdots,t; j=1,2,\cdots,l_t$. Let $c_{r_i,j}$ be

$$c_{r_i,j}=c_{i,j,0}x^{d_{ij}}+\cdots+c_{i,j,d_{ij}}. \tag{14}$$

Definition 8.1 The matrix $\mathcal{C}$ in (13) is called a generalized Hermite normal form if it satisfies the following conditions:

1) $0\le d_{r_i,1}<d_{r_i,2}<\cdots<d_{r_i,l_i}$ for any i.

2) $c_{r_i,l_i,0}|\cdots|c_{r_i,2,0}|c_{r_i,1,0}$.

3) $S(\boldsymbol{c}_{r_i,j_1},\boldsymbol{c}_{r_i,j_2}) = x^{d_{r_i,j_2}-d_{r_i,j_1}}\boldsymbol{c}_{r_i,j_1} - \frac{c_{r_i,j_1,0}}{c_{r_i,j_2,0}}\boldsymbol{c}_{r_i,j_2}$ can be reduced to zero by the column vectors of the matrix for any $1 \le i \le t, 1 \le j_1 < j_2 \le l_i$.

4) $\boldsymbol{c}_i$ is reduced w.r.t. the column vectors of the matrix other than $\boldsymbol{c}_i$, for any $1 \le i \le s$.

It is proved in [16] that $\mathbb{f} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_s\} \subseteq \mathbb{Z}[x]^n$ is a reduced Gröbner basis such that $\boldsymbol{f}_1 < \boldsymbol{f}_2 < \cdots < \boldsymbol{f}_s$ if and only if the matrix $[\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_s]$ is a generalized Hermite normal form. From [17], generalized Hermite normal forms can be computed in polynomial-time.

For $S \subseteq \mathbb{Z}[x]^n$, we use $(S)_D$ to denote the D-module generated by S in D^n, where $D = \mathbb{Z}[x]$ or $D = \mathbb{Q}[x]$. When $D = \mathbb{Z}[x]$, $(S)_D$ is the $\mathbb{Z}[x]$-lattice generated by S. Similarly, let A be a matrix with entries in $\mathbb{Z}[x]$. We use $(A)_D$ to denote the D-module generated by the column vectors of A.

A $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^n$ is said to be $\mathbb{Z}$-saturated if, for any $a \in \mathbb{Z}^*$ and $\boldsymbol{f} \in \mathbb{Z}[x]^n$, $a\boldsymbol{f} \in L$ implies $\boldsymbol{f} \in L$. The $\mathbb{Z}$-saturation of L is defined to be

$$\mathrm{sat}_{\mathbb{Z}}(L) = \{\boldsymbol{f} \in \mathbb{Z}[x]^n \mid \exists a \in \mathbb{Z}^* \text{ s.t. } a\boldsymbol{f} \in L\}.$$

We need the following algorithm from [16].

• **ZFactor**($\mathcal{C}$): for a generalized Hermite normal form $\mathcal{C}$, the algorithm returns $\emptyset$ if $L = (\mathcal{C})_{\mathbb{Z}[x]}$ is $\mathbb{Z}$-saturated, or a finite set $S \subseteq \mathrm{sat}_{\mathbb{Z}}(L) \setminus (L)$.

The following algorithm checks whether $L = (\mathcal{C})_{\mathbb{Z}[x]}$ is $\mathbb{Z}[x]$-saturated and in the negative case returns elements of $\mathrm{sat}_{\mathbb{Z}[x]}(L) \setminus L$.

Algorithm 1 — ZXFactor($\mathcal{C}$)

Input A generalized Hermite normal form $\mathcal{C} \in \mathbb{Z}[x]^{n\times s}$ given in (13).

Output $\emptyset$, if $L = (\mathcal{C})_{\mathbb{Z}[x]}$ is $\mathbb{Z}[x]$-saturated; otherwise, a finite set $\{\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots, \boldsymbol{h}_r\} \subseteq \mathbb{Z}[x]^n$ such that $\boldsymbol{h}_i \in \mathrm{sat}_{\mathbb{Z}[x]}(L) \setminus L$, $i = 1, 2, \cdots, r$.

1) Let $S =$**ZFactor**($\mathcal{C}$). If $S \neq \emptyset$ return S.

2) For any prime factor $p(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$ of $\prod_{i=1}^{t} c_{r_i,1}$, execute Steps 2.1)–2.3), where $c_{r_i,1}$ are from (13).

2.1) Set $M = [\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_t,1}] \in \mathbb{Z}[x]^{n\times t}$, where $\boldsymbol{c}_{r_i,1}$ can be found in (13).

2.2) Compute a finite basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_l\}$ of $\mathrm{Ker}(M) = \{X \in \mathbb{Q}[x]^t \mid MX = \boldsymbol{0}\}$ as a K-vector space in K^t, where $K = \mathbb{Q}[x]/(p(x))$.

2.3) If $B \neq \emptyset$,

2.3.1) For each $\boldsymbol{b}_i$, let $M\boldsymbol{b}_i = p(x)\frac{\boldsymbol{g}_i}{m_i}$, where $\boldsymbol{g}_i \in \mathbb{Z}[x]^n$ and $m_i \in \mathbb{Z}$.

2.3.2) Return $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_l\}$

3) Return $\emptyset$.

The $\mathbb{Z}[x]$-saturation of a $\mathbb{Z}[x]$-lattice L is defined to be

$$\mathrm{sat}_{\mathbb{Z}[x]}(L) = \{\boldsymbol{f} \in \mathbb{Z}[x]^n \mid \exists p \in \mathbb{Z}[x]\backslash\{0\} \text{ s.t. } p\boldsymbol{f} \in L\}.$$

The following algorithm compute $\mathrm{sat}_{\mathbb{Z}[x]}(L)$.

Algorithm 2 — SatZX$(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m)$

Input A finite set $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\} \subseteq \mathbb{Z}[x]^n$.

Output A set of generators of $\mathrm{sat}_{\mathbb{Z}[x]}(L)$, where $L = (U)_{\mathbb{Z}[x]}$.

1) Compute a generalized Hermite normal form $\mathbb{g}$ of U [17].

2) Set $S =$**ZXFactor**$(\mathbb{g})$.

3) If $S = \emptyset$, return $\mathbb{g}$; otherwise set $U = \mathbb{g} \cup S$ and go to Step 1).

Example 8.2 Let

$$\mathcal{C} = \begin{bmatrix} x & x^2+1 \\ 2x^2+1 & 0 \\ 0 & 4x^2+2 \end{bmatrix}.$$

Apply Algorithm **ZXFactor** to $\mathcal{C}$. In Step 1), $S = \emptyset$ and $\mathcal{C}$ is $\mathbb{Z}$-saturated. In Step 2), the only irreducible factor of $\prod_{i=1}^{t} c_{r_i,1} \in \mathbb{Z}[x]$ is $p(x) = 2x^2+1$. In Step 2.1), $M = \mathcal{C}$ and in Step 2.2), $B = \{[-1, 2x]^\tau\}$. In Step 2.3.1), $M \cdot [-1, 2x]^\tau = 2x\boldsymbol{c}_{2,1} - \boldsymbol{c}_{1,1} = p(x)[x, -1, 4x]^\tau = 0 \bmod p(x)$ and $\{[x, -1, 4x]^\tau\}$ is returned.

In Algorithm **SatZX**, $\boldsymbol{h} = [x, -1, 4x]^\tau$ is added into $\mathcal{C}$ and the generalized Hermite normal form of $\mathcal{C} \cup \{\boldsymbol{h}\}$ is

$$\mathcal{C}_1 = \begin{bmatrix} x & 1 \\ 2x^2+1 & x \\ 0 & 2 \end{bmatrix}.$$

Apply Algorithm **ZXFactor** to $\mathcal{C}_1$, one can check that $\mathcal{C}_1$ is $\mathbb{Z}[x]$-saturated.

In the rest of this section, we will prove the correctness of the algorithm. As with the definition of $\mathrm{sat}_{\mathbb{Z}[x]}(L)$, we can define $\mathrm{sat}_{\mathbb{Q}[x]}(L_{\mathbb{Q}[x]})$. $L_{\mathbb{Q}[x]}$ is said to be $\mathbb{Q}[x]$-saturated if $\mathrm{sat}_{\mathbb{Q}[x]}(L_{\mathbb{Q}[x]}) = L_{\mathbb{Q}[x]}$. The following lemma gives a criterion for whether L is $\mathbb{Z}[x]$-saturated.

Lemma 8.3 *A $\mathbb{Z}[x]$-lattice L is $\mathbb{Z}[x]$-saturated if and only if* $\mathrm{sat}_{\mathbb{Z}}(L) = L$ *and* $\mathrm{sat}_{\mathbb{Q}[x]}(L_{\mathbb{Q}[x]}) = L_{\mathbb{Q}[x]}$.

Proof " $\Rightarrow$ ". If $L = (\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_m\})_{\mathbb{Z}[x]}$ is $\mathbb{Z}[x]$-saturated, then $\mathrm{sat}_{\mathbb{Z}}(L) = L$. If $\mathrm{sat}_{\mathbb{Q}[x]}(L_{\mathbb{Q}[x]}) \neq L_{\mathbb{Q}[x]}$, then there exist an $h(x) \in \mathbb{Q}[x]$ and a $\boldsymbol{g} \in \mathbb{Q}[x]^n$, such that $h(x)\boldsymbol{g} \in L_{\mathbb{Q}[x]}$ but $\boldsymbol{g} \notin L_{\mathbb{Q}[x]}$. From $h(x)\boldsymbol{g} \in L_{\mathbb{Q}[x]}$, we have $h(x)\boldsymbol{g} = \sum_{i=1}^{s} q_i(x)\boldsymbol{u}_i$, where $q_i(x) \in \mathbb{Q}[x]$. Clearing the denominators of the above equation, there exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 h(x) \in \mathbb{Z}[x]$, $m_2\boldsymbol{g} \in \mathbb{Z}[x]^n$, and $m_1 h(x) \cdot m_2 \boldsymbol{g} \in L$. Since L is $\mathbb{Z}[x]$-saturated, $m_2\boldsymbol{g} \in L$, which contradicts to $\boldsymbol{g} \notin L_{\mathbb{Q}[x]}$.

" $\Leftarrow$ ". For any $h(x) \in \mathbb{Z}[x]$ and $\boldsymbol{g} \in \mathbb{Z}[x]^n$, if $h(x)\boldsymbol{g} \in L$, we have $h(x)\boldsymbol{g} \in L_{\mathbb{Q}[x]}$, and hence $\boldsymbol{g} \in L_{\mathbb{Q}[x]}$ since $\mathrm{sat}_{\mathbb{Q}[x]}(L_{\mathbb{Q}[x]}) = L_{\mathbb{Q}[x]}$. Since $\boldsymbol{g} \in L_{\mathbb{Q}[x]}$, there exists an $m \in \mathbb{Z}$ such that $m\boldsymbol{g} \in L$, which implies $\boldsymbol{g} \in L$ since L is $\mathbb{Z}$-saturated. ∎

In the following two lemmas, $\mathcal{C}$ is the generalized Hermite normal form given in (13).

Lemma 8.4 $(\mathcal{C})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_t,1})_{\mathbb{Q}[x]}$.

Proof We will prove $(\mathcal{C})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_t,1})_{\mathbb{Q}[x]}$ by induction. By 3) of Definition 8.1, $S(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_1,2}) = x^u \boldsymbol{c}_{r_1,1} - a\boldsymbol{c}_{r_1,2}$ ($u \in \mathbb{N}$ and $a \in \mathbb{Z}$) can be reduced to zero by $\boldsymbol{c}_{r_1,1}$, which means $\boldsymbol{c}_{r_1,2} = q(x)\boldsymbol{c}_{r_1,1}$ where $q(x) \in \mathbb{Q}[x]$. Hence, $(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_1,2})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1})_{\mathbb{Q}[x]}$ as $\mathbb{Q}[x]$-modules. Suppose for $k < l_1$, $(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_1,k})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1})_{\mathbb{Q}[x]}$ as $\mathbb{Q}[x]$-modules. We will show that $(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_1,k+1})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1})_{\mathbb{Q}[x]}$ as $\mathbb{Q}[x]$-modules. Indeed, by 3) of Definition 8.1, $S(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_1,k+1}) = x^v \boldsymbol{c}_{r_1,1} - b\boldsymbol{c}_{r_1,k+1}$ ($v \in \mathbb{N}$ and $b \in \mathbb{Z}$) can be reduced to zero by $\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_1,k}$ and hence, $\boldsymbol{c}_{r_1,k+1} \in (\boldsymbol{c}_{r_1,1})_{\mathbb{Q}[x]}$. Then we have $(\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_1,l_1})_{\mathbb{Q}[x]} = (\boldsymbol{c}_{r_1,1})_{\mathbb{Q}[x]}$. For the rest of the polynomials in $\mathcal{C}$, the proof is similar. ∎

The following lemma gives a criterion for a $\mathbb{Q}[x]$-module to be $\mathbb{Q}[x]$-saturated.

Lemma 8.5 *Let* $L = (\mathcal{C})_{\mathbb{Q}[x]}$. *Then* $\mathrm{sat}_{\mathbb{Q}[x]}(L) = L$ *if and only if* $\mathcal{C}_1 = \{\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_t,1}\}$ *is linear independent over* $\mathbb{K}_{p(x)} = \mathbb{Q}[x]/(p(x))$ *for any irreducible polynomial* $p(x) \in \mathbb{Z}[x]$.

Proof " $\Rightarrow$ ". Assume the contrary, that is, $\mathcal{C}_1$ are linear dependent over $\mathbb{K}_{p(x)}$ for some $p(x)$. Then there exist $g_i \in \mathbb{Q}[x]$ not all zero in $\mathbb{K}_{p(x)}$, such that $\sum_{i=1}^t g_i \boldsymbol{c}_{r_i,1} = \boldsymbol{0}$ in $\mathbb{K}_{p(x)}^n$ and hence $\sum_{i=1}^t g_i \boldsymbol{c}_{r_i,1} = p(x)\boldsymbol{g}$ in $\mathbb{Q}[x]^n$. Since $\mathcal{C}_1$ is in upper triangular form and is clearly linear independent in $\mathbb{Q}[x]^n$, we have $\boldsymbol{g} \neq \boldsymbol{0}$. Since $\mathrm{sat}_{\mathbb{Q}[x]}(L) = L$, we have $\boldsymbol{g} \in L$. Then, there exist $f_i \in \mathbb{Q}[x]$ such that $\boldsymbol{g} = \sum_{i=1}^t f_i \boldsymbol{c}_{r_i,1}$. Hence $\sum_{i=1}^t (g_i - pf_i)\boldsymbol{c}_{r_i,1} = \boldsymbol{0}$ in $\mathbb{Q}[x]^n$. Since $\mathcal{C}_1$ is linear independent in $\mathbb{Q}[x]^n$, $g_i = pf_i$ and hence $g_i = 0$ in $\mathbb{K}_{p(x)}$, a contradiction.

" $\Leftarrow$ ". Assume the contrary, that is, there exists $\boldsymbol{g} \in \mathbb{Q}[x]^n$, such that $\boldsymbol{g} \notin L$ and $p(x)\boldsymbol{g} \in L$ for an irreducible polynomial $p(x) \in \mathbb{Z}[x]$. Then by Lemma 8.4, we have $p\boldsymbol{g} = \sum_{i=1}^t f_i \boldsymbol{c}_{r_i,1}$, where $f_i \in \mathbb{Q}[x]$. p cannot be a factor of all f_i. Otherwise, $\boldsymbol{g} = \sum_{i=1}^t \frac{f_i}{p} \boldsymbol{c}_{r_i,1} \in L$. Then some of f_i is not zero in $\mathbb{K}_{p(x)}$, which means $\sum_{i=1}^t f_i \boldsymbol{c}_{r_i,1} = \boldsymbol{0}$ is a nontrivial linear relation among $\mathcal{C}_1$ over $\mathbb{K}_{p(x)}$, a contradiction. ∎

From the " $\Rightarrow$ " part of the above proof, we have the following corollary.

Corollary 8.6 *Let* $\mathcal{C}$ *be the generalized Hermite normal form given in* (13) *and* $\sum_{i=1}^t f_i \boldsymbol{c}_{r_i,1} = 0$ *a nontrivial linear relation among* $\boldsymbol{c}_{r_i,1}$ *in* $(\mathbb{Q}[x]/(p(x)))^n$, *where* $p(x)$ *is an irreducible polynomial in* $\mathbb{Z}[x]$ *and* $f_i \in \mathbb{Q}[x]$. *Then, in* $\mathbb{Q}[x]^n$, $\sum_{i=1}^r f_i \boldsymbol{c}_{r_i,1} = p(x)\boldsymbol{g}$ *and* $\boldsymbol{g} \notin (\mathcal{C})_{\mathbb{Q}[x]}$.

Theorem 8.7 *Algorithms* **SatZX** *and* **ZXFactor** *are correct.*

Proof In Step 3) of Algorithm **SatZX**, if $(\mathrm{g})_{\mathbb{Z}[x]}$ is not $\mathbb{Z}[x]$-saturated, then $S \neq \emptyset$ and $(\mathrm{g})_{\mathbb{Z}[x]} \subsetneq (\mathrm{g} \cup S)_{\mathbb{Z}[x]} \subseteq \mathrm{sat}_{\mathbb{Z}[x]}((\mathrm{g})_{\mathbb{Z}[x]})$. Since $\mathbb{Z}[x]^n$ is Notherian, the algorithm will terminate and output $\mathrm{sat}_{\mathbb{Z}[x]}(L)$. Thus, it suffices to prove the correctness of Algorithm **ZXFactor**.

In Step 1) of Algorithm **ZXFactor**, if $S \neq \emptyset$, then from properties of Algorithm **ZFactor**, $S \subseteq \mathrm{sat}_{\mathbb{Z}}((\mathcal{C})_{\mathbb{Z}[x]}) \setminus (\mathcal{C})_{\mathbb{Z}[x]} \subseteq \mathrm{sat}_{\mathbb{Z}[x]}((\mathcal{C})_{\mathbb{Z}[x]}) \setminus (\mathcal{C})_{\mathbb{Z}[x]}$. The algorithm is correct. In Step 2), we claim that L is $\mathbb{Q}[x]$-saturated if and only if $B = \emptyset$, and if $B \neq \emptyset$ then $\boldsymbol{g}_i$ in Step 2.3.1) is not in L. In Step 3), L is both $\mathbb{Z}$- and $\mathbb{Q}[x]$-saturated. By Lemma 8, L is $\mathbb{Z}[x]$-saturated and the algorithm is correct. So, it suffices to prove the claim about Step 2).

Let $L = (\mathcal{C})_{\mathbb{Z}[x]}$. In Step 2), L is already $\mathbb{Z}$-saturated. Then by Lemma 8.3, L is $\mathbb{Z}[x]$-saturated if and only if $(\mathcal{C})_{\mathbb{Q}[x]}$ is $\mathbb{Q}[x]$-saturated. By Lemma 8.5, to check whether $(\mathcal{C})_{\mathbb{Q}[x]}$ is $\mathbb{Q}[x]$-saturated, we only need to check whether for any irreducible polynomial $p(x) \in \mathbb{Z}[x]$, $\mathcal{C}_1 = \{\boldsymbol{c}_{r_1,1}, \boldsymbol{c}_{r_2,1}, \cdots, \boldsymbol{c}_{r_t,1}\}$ is linear independent over $\mathbb{K}_{p(x)} = \mathbb{Q}[x]/(p(x))$. If $p(x)$ is not a prime factor of $\prod_{i=1}^{t} c_{r_i,1}$, then the leading monomials of $\boldsymbol{c}_{r_i,1}, i = 1, 2, \cdots, t$ are nonzero and $\mathcal{C}_1$ is in upper triangular form. As a consequence, $\mathcal{C}_1$ must be linear independent over $\mathbb{K}_{p(x)}$. Then, in order to check whether L is $\mathbb{Q}[x]$-saturated, it suffices to consider the prime factors of $\prod_{i=1}^{t} c_{r_i,1}$ in Step 2) of the algorithm. In Step 2.3), it is clear that if $B = \emptyset$ then $\mathcal{C}_1$ is linear independent over $\mathbb{K}_{p(x)}$. For $\boldsymbol{b}_i \in B$, since $M\boldsymbol{b}_i = \boldsymbol{0}$ over $\mathbb{K}_{p(x)}$, $M\boldsymbol{b}_i = p(x)\boldsymbol{h}_i$ where $\boldsymbol{h}_i \in \mathbb{Q}[x]^t$. Hence $\boldsymbol{h}_i = \frac{\boldsymbol{g}_i}{m_i}$ for $\boldsymbol{g}_i \in \mathbb{Z}[x]^t$ and $m_i \in \mathbb{Z}$. By Corollary 8.6, $\boldsymbol{g}_i \notin L$. Therefore, Step 2) returns a set of nontrivial factors of L if L is not $\mathbb{Z}[x]$-saturated. The claim about Step 2) is proved. ▌

9 Conclusion

In this paper, we initiate the study of toric σ-varieties. A toric σ-variety is defined as the Cohn closure of the values of a set of Laurent σ-monomials. Three characterizing properties of toric σ-varieties are proved in terms of their coordinate rings, their defining ideals, and group actions. In particular, a σ-variety is toric if and only if its defining ideal is a toric σ-ideal, meaning a binomial σ-ideal whose support lattice is $\mathbb{Z}[x]$-saturated. Algorithms are given to decide whether the binomial σ-ideal $\mathcal{I}_L$ with the support lattice L is toric.

We establish connections between toric σ-varieties and affine $\mathbb{N}[x]$-semimodules. We show that the category of toric σ-varieties with toric morphisms is antiequivalent to the category of affine $\mathbb{N}[x]$-semimodules with $\mathbb{N}[x]$-semimodule morphisms. We also show that there is a one-to-one correspondence between the irreducible T-invariant σ-subvarieties of a toric σ-variety X and the faces of the corresponding affine $\mathbb{N}[x]$-semimodule, where T is the σ-torus of X. Besides, there is also a one-to-one correspondence between the T-orbits of a toric σ-variety X and the faces of the corresponding affine $\mathbb{N}[x]$-semimodule S, when S is face-saturated.

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