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Toric P-difference varieties

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Abstract In this paper, we introduce the concept of P-difference varieties and study the properties of toric P-difference varieties. Toric P-difference varieties are analogues of toric varieties in difference algebraic geometry. The category of affine toric P-difference varieties with toric morphisms is shown to be antiequivalent to the category of affine $P[x]$-semimodules with $P[x]$-semimodule morphisms. Moreover, there is a one-to-one correspondence between the irreducible invariant P-difference subvarieties of an affine toric P-difference variety and the faces of the corresponding affine $P[x]$-semimodule. We also define abstract toric P-difference varieties by gluing affine toric P-difference varieties. The irreducible invariant P-difference subvarieties-faces correspondence is generalized to abstract toric P-difference varieties. By virtue of this correspondence, a divisor theory for abstract toric P-difference varieties is developed.

Keywords $\mathbb{Z}[x]$-lattice, affine $P[x]$-semimodule, P-difference variety, toric P-difference variety, difference torus

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1 Introduction

Difference algebra and difference algebraic geometry [1, 9, 12, 20] were founded by Ritt and Doob [18], and Cohn [1], who aimed to study algebraic difference equations as algebraic geometry to polynomial equations. The motivation of this paper is to develop a divisor theory for toric difference varieties.

Toric varieties are very important objects of study in algebraic geometry, since they have deep connections with the theory of polytopes, symplectic geometry and mirror symmetry, and have applications in many other fields such as physics, coding theory, algebraic statistics and geometric modeling [3, 4, 7, 17]. Toric difference varieties are analogues of toric varieties in difference algebraic geometry and were first studied by Gao, Huang, Wang and Yuan in [5], since toric difference varieties establish a connection between difference Chow forms [14] and sparse difference resultants [13].

Simply speaking, an affine toric difference variety is an affine difference variety which can be parameterized by monomials, or equivalently, is an irreducible affine difference variety containing a difference torus as a Cohn open subset such that the group action of the difference torus on itself extends to a difference algebraic group action on the affine difference variety. In the paper [5], many basic properties of affine toric difference varieties were characterized in terms of affine $\mathbb{N}[x]$-semimodules. Actually, the

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category of affine toric difference varieties with toric morphisms is antiequivalent to the category of affine $\mathbb{N}[x]$-semimodules with $\mathbb{N}[x]$-semimodule morphisms.

In algebraic geometry, the divisor theory is a very useful tool to study the properties of algebraic varieties [3, 8]. However, the divisor theory for toric difference varieties defined in [5] does not behave well. The main obstacle is that we cannot define valuation maps on the difference coordinate rings of these toric difference varieties. To overcome this obstacle and develop a divisor theory for toric difference varieties, we will introduce a new generalization of toric varieties in difference algebraic geometry, i.e. toric P-difference varieties.

The notion of P-difference varieties is a generalization of the notion of classical difference varieties by admitting variables of the defining difference polynomials with negative degrees in some sense. More concretely, we define an order on $\mathbb{Z}[x]$ as follows: $f = \sum_{i=0}^{p} a_i x^i > g = \sum_{i=0}^{p} b_i x^i$ if and only if there exists an integer s such that $a_i = b_i$ for $s+1 \leqslant i \leqslant p$ and $a_s > b_s$. Let $P[x] = \{f \in \mathbb{Z}[x] \mid f \geqslant 0\}$. For $g = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]$, we write $a^g = \prod_{i=1}^{s} (\sigma^i(a))^{c_i}$ for $a \neq 0$ in a difference field, i.e., a field equipped with an endomorphism σ, and set $0^g = 0$ for $g \in P[x] \setminus \{0\}$. The $P[\sigma]$-polynomial ring $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ over a difference field k is the polynomial ring in the variables $y_i^g, i = 1, \ldots, m, g \in P[x] \setminus \{0\}$, endowed with a difference algebra structure. An element in $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is called a P-difference polynomial. An affine P-difference variety over k is the zero sets defined by a set of P-difference polynomials, which is a functor from the category of difference field extensions of k to the category of sets. Now we can say that an affine toric P-difference variety is an affine P-difference variety which can be parameterized by monomials. As in the algebraic case, there is a difference algebraic group action on an affine toric P-difference variety. Actually, an affine toric P-difference variety is an irreducible affine P-difference variety containing a difference torus as an open subset such that the action of the difference torus on itself extends to a difference algebraic group action on the affine P-difference variety.

Every affine toric P-difference variety corresponds to an affine $P[x]$-semimodule. It turns out that many properties of affine toric P-difference varieties can be described in terms of affine $P[x]$-semimodules. Actually, the category of affine toric P-difference varieties with toric morphisms is antiequivalent to the category of affine $P[x]$-semimodules with $P[x]$-semimodule morphisms. Since there is a difference algebraic group action of the difference torus on a toric P-difference variety, we obtain irreducible invariant P-difference subvarieties and orbits under the action of the difference torus. We establish a one-to-one correspondence between the irreducible invariant P-difference subvarieties of an affine toric P-difference variety and the faces of the corresponding affine $P[x]$-semimodule and a one-to-one correspondence between the difference torus orbits of an affine toric difference variety and the faces of the corresponding affine $P[x]$-semimodule.

A fan is defined to be a finite set of affine $P[x]$-semimodules which satisfies certain compatible conditions. We further define the abstract toric P-difference variety associated with a fan by gluing affine toric P-difference varieties along open subsets. As examples, projective toric P-difference varieties defined by $\mathbb{Z}[x]$-lattice points are abstract toric P-difference varieties. The irreducible invariant P-difference subvarieties-faces correspondence still applies to abstract toric P-difference varieties constructed from fans. By virtue of this correspondence, we can define divisors and divisor class modules for toric P-difference varieties. In particular, the class module and the Picard module of a toric P-difference variety are defined. Moreover, we establish connections between the properties of toric P-difference varieties and divisor class modules.

The rest of this paper is organized as follows. In Section 2, we list some preliminaries for difference algebraic geometry and preliminaries for $\mathbb{Z}[x]$-lattices which will be used in this paper. In section 3, we introduce the concept of P-difference varieties. In section 4, affine toric difference varieties are defined and basic properties are proved. In section 5, projective toric difference varieties are defined and basic properties are proved. In section 6, we will define abstract toric difference varieties associated with fans and prove their basic properties. In section 7, we will develop a divisor theory for toric P-difference varieties. Conclusions are given in Section 8.

2 Preliminaries

We list some basic notations and results on difference algebraic geometry and $\mathbb{Z}[x]$-lattices in this section. For more details about difference algebraic geometry, please refer to [9, 20]. For more details about $\mathbb{Z}[x]$-lattices, please refer to [5, 11].

2.1 Preliminaries for Difference Algebraic Geometry

First we recall some basic notions from difference algebra. For more details, please refer to [12, 20]. All rings in this paper will be assumed to be commutative and unital.

A *difference ring* or *σ-ring* for short (R, σ), is a ring R together with a ring endomorphism $\sigma\colon R \to R$. If R is a field, then we call it a *difference field*, or a *σ-field* for short. We usually omit σ from the notation, simply refer to R as a σ-ring or a σ-field. A typical example of σ-field is the field of rational functions $\mathbb{Q}(x)$ with $\sigma(f(x)) = f(x+1)$. A *morphism between σ-rings* R and S is a ring homomorphism $\psi\colon R \to S$ which commutes with σ. In this paper, all σ-fields will be assumed to be of characteristic 0 and k is always assumed to be a fixed σ-field.

A k-algebra R is called a *k-σ-algebra* if the algebra structure map $k \to R$ is a morphism of σ-rings. A *morphism of k-σ-algebras* is a morphism of k-algebras which is also a morphism of σ-rings. A k-subalgebra of a k-σ-algebra is called a *k-σ-subalgebra* if it is closed under σ. If a k-σ-algebra is a σ-field, then it is called a *σ-field extension* of k. Let R and S be two k-σ-algebras. Then $R \otimes_k S$ is naturally a k-σ-algebra by defining $\sigma(r \otimes s) = \sigma(r) \otimes \sigma(s)$ for $r \in R$ and $s \in S$.

Let R be a k-σ-algebra. For a subset A of R, the smallest k-σ-subalgebra of R containing A is denoted by $k\{A\}$. If there exists a finite subset A of R such that $R = k\{A\}$, we say that R is *finitely σ-generated* over k. If additionally R is a σ-field, the smallest k-σ-subfield of R containing A is denoted by $k\langle A\rangle$.

Now we introduce the following useful notation. Let x be an algebraic indeterminate and $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{Z}[x]$. Let K be any σ-field. Then for $a \in K^*$, we write $a^p = \prod_{i=0}^{s} (\sigma^i(a))^{c_i}$. It is easy to check that for $p, q \in \mathbb{Z}[x]$, we have $a^{p+q} = a^p a^q, a^{pq} = (a^p)^q$.

Let R be a σ-ring. A *σ-ideal* I in R is an algebraic ideal which is closed under σ, i.e. $\sigma(I) \subseteq I$. If I also has the property that $\sigma(a) \in I$ implies $a \in I$, then it is called a *reflexive* σ-ideal. A *σ-prime* ideal is a reflexive σ-ideal which is prime as an algebraic ideal. A σ-ideal I is called a *perfect* σ-ideal if for any $g \in \mathbb{N}[x] \setminus \{0\}$ and $a \in R$, $a^g \in I$ implies $a \in I$. It is easy to prove that every σ-prime ideal is perfect. If S is a subset of R, then we use (S), $[S]$, and $\{S\}$ to denote the algebraic ideal, the σ-ideal, and the perfect σ-ideal generated by S respectively.

Suppose that $\mathbb{Y} = \{y_1, \ldots, y_m\}$ is a set of σ-indeterminates over k (which means that all the variables $\mathbb{Y}, \sigma(\mathbb{Y}), \sigma^2(\mathbb{Y}), \ldots$ are algebraicly independent over k). Then the *σ-polynomial ring* over k in $\mathbb{Y}$ is the polynomial ring in the variables $\mathbb{Y}, \sigma(\mathbb{Y}), \sigma^2(\mathbb{Y}), \ldots$. It is denoted by $k\{y_1, \ldots, y_m\}$ and has a natural k-σ-algebra structure. An element in $k\{y_1, \ldots, y_m\}$ is called a *σ-polynomial* over k.

For convenience, we denote the category of σ-field extensions of k by $\mathscr{E}_k$ and write $K \in \mathscr{E}_k$ to express that K is a σ-field extension of k. Let $F \subseteq k\{y_1, \ldots, y_m\}$ be a set of σ-polynomials. For any $K \in \mathscr{E}_k$, define the solutions of F in K to be

$$\mathbb{V}_K(F) := \{a \in K^m \mid f(a) = 0 \text{ for all } f \in F\}.$$

Note that $K \rightsquigarrow \mathbb{V}_K(F)$ is naturally a functor from the category of σ-field extensions of k to the category of sets. Let us denote this functor by $\mathbb{V}(F)$.

Definition 2.1. An *(affine) difference variety* or *σ-variety* over k is a functor X from the category of σ-field extensions of k to the category of sets which is of the form $\mathbb{V}(F)$ for some subset F of $k\{y_1, \ldots, y_m\}$. In this situation, we say that X is the (affine) σ-variety defined by F.

If no confusion is caused, we will omit the word "affine" for simplicity.

The functor $\mathbb{A}_k^m$ given by $\mathbb{A}_k^m(K) = K^m$ for $K \in \mathscr{E}_k$ is called the *σ-affine (m-)space* over k. Obviously, $\mathbb{A}_k^m = \mathbb{V}(0)$ is an affine σ-variety over k.

If X and Y are two σ-varieties over k, then we write $X \subseteq Y$ to indicate that X is a subfunctor of Y. This simply means $X(K) \subseteq Y(K)$ for every $K \in \mathscr{E}_k$. In this situation, we also say that X is a *σ-subvariety* of Y.

Let X be a σ-subvariety of $\mathbb{A}_k^m$. Then

$$\mathbb{I}(X) := \{f \in k\{y_1, \ldots, y_m\} \mid f(a) = 0 \text{ for all } a \in X(K) \text{ and all } K \in \mathscr{E}_k\}$$

is called the *defining ideal* or the *vanishing ideal* of X, which is a perfect σ-ideal. In [20, Chapter 2], it is proved that the σ-subvarieties of $\mathbb{A}_k^m$ are in a one-to-one correspondence with the perfect σ-ideals of $k\{y_1, \ldots, y_m\}$ and $\mathbb{I}(\mathbb{V}(F)) = \{F\}$ for $F \subseteq k\{y_1, \ldots, y_m\}$.

Definition 2.2. Let X be a σ-subvariety of $\mathbb{A}_k^m$. Then the k-σ-algebra

$$k\{X\} := k\{y_1, \ldots, y_m\}/\mathbb{I}(X)$$

is called the *σ-coordinate ring* of X.

A k-σ-algebra is called an *affine k-σ-algebra* if it is isomorphic to $k\{y_1, \ldots, y_m\}/I$ for some perfect σ-ideal I.

The following lemma is taken from [20, Remark 2.1.10].

Lemma 2.3. *Let X be a k-σ-variety. Then for any $K \in \mathscr{E}_k$, there is a natural bijection between $X(K)$ and the set of k-σ-algebra morphisms from $k\{X\}$ to K. Indeed,*

$$X \simeq \mathrm{Hom}(k\{X\}, -)$$

as functors.

Definition 2.4. Let $X \subseteq \mathbb{A}_k^m$ and $Y \subseteq \mathbb{A}_k^n$ be k-σ-varieties. A morphism of functors $f\colon X \to Y$ is called a *morphism of k-σ-varieties* if there exist σ-polynomials $f_1, \ldots, f_n \in k\{y_1, \ldots, y_m\}$ such that $f(a) = (f_1(a), \ldots, f_n(a))$ for every $a \in X(K)$ and all $K \in \mathscr{E}_k$.

In analogy with affine algebraic varieties, we have

Theorem 2.5. *The category of affine k-σ-varieties is antiequivalent to the category of affine k-σ-algebras.*

Proof. Please refer to [20, Theorem 2.1.21]. □

Suppose that X is an affine k-σ-variety. Let $\mathrm{Spec}^\sigma(k\{X\})$ be the set of all σ-prime ideals of $k\{X\}$, which is called the *σ-spectrum* of $k\{X\}$. Let $F \subseteq k\{X\}$. Set

$$\mathcal{V}(F) := \{\mathfrak{p} \in \mathrm{Spec}^\sigma(k\{X\}) \mid F \subseteq \mathfrak{p}\} \subseteq \mathrm{Spec}^\sigma(k\{X\}).$$

Obviously, $\mathcal{V}(F) = \mathcal{V}(\{F\})$. It can be checked that $\mathrm{Spec}^\sigma(k\{X\})$ is a topological space with closed sets of the forms $\mathcal{V}(F), F \subseteq k\{X\}$. Then the *topological space* of X is $\mathrm{Spec}^\sigma(k\{X\})$ equipped with the above Cohn topology.

Let $F \subseteq k\{y_1, \ldots, y_m\}$ and $K, L \in \mathscr{E}_k$. Two solutions $a \in \mathbb{V}_K(F)$ and $b \in \mathbb{V}_L(F)$ are said to be *equivalent* if there exists a k-σ-isomorphism between $k\langle a\rangle$ and $k\langle b\rangle$ which maps a to b. Obviously this defines an equivalence relation. The following theorem gives a relationship between equivalence classes of solutions of F and σ-prime ideals containing F. For the proof, please refer to [20, Theorem 2.2.1].

Theorem 2.6. *Let X be a k-σ-variety. There is a natural bijection between the set of equivalence classes of solutions of $\mathbb{I}(X)$ and $\mathrm{Spec}^\sigma(k\{X\})$.*

Because of Theorem 2.6, we shall not strictly distinguish between a σ-variety and its topological space. In other words, we use X to mean the σ-variety or its topological space.

2.2 Preliminaries for $\mathbb{Z}[x]$-lattices

A $\mathbb{Z}[x]$-module that can be embedded into $\mathbb{Z}[x]^n$ for some n is called a $\mathbb{Z}[x]$-*lattice*. Since $\mathbb{Z}[x]^n$ is Noetherian as a $\mathbb{Z}[x]$-module, we see that any $\mathbb{Z}[x]$-lattice is finitely generated. Let L be a $\mathbb{Z}[x]$-lattice.

We always identify it with a $\mathbb{Z}[x]$-submodule of $\mathbb{Z}[x]^n$ for some n. Define the *rank* of L to be

$$\operatorname{rank}(L) := \dim_{\mathbb{Q}(x)} L \otimes \mathbb{Q}(x).$$

Note that L may not be a free $\mathbb{Z}[x]$-module, thus the number of minimal generators of L can be larger than its rank.

Sometimes we want to know whether a $\mathbb{Z}[x]$-module is a $\mathbb{Z}[x]$-lattice, i.e. whether it can be embedded into $\mathbb{Z}[x]^n$ for some n. The following lemma is taken from [19, p.172]:

Lemma 2.7. *Let R be a domain and A an R-module. If A is finitely generated and torsion-free, then A can be imbedded into a finitely generated free R-module.*

Therefore, the condition for a finitely generated $\mathbb{Z}[x]$-module to be a $\mathbb{Z}[x]$-lattice is that it has no torsion.

Suppose $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$. The *syzygy module* of U, which is denoted by $\operatorname{Syz}(U)$, is

$$\operatorname{Syz}(U) := \{\mathbf{v} \in \mathbb{Z}[x]^m \mid U\mathbf{v} = \mathbf{0}\},$$

where we regard U as a matrix with columns $\mathbf{u}_i, i = 1, \ldots, m$.

It is clear that $\operatorname{Syz}(U)$ is a $\mathbb{Z}[x]$-lattice in $\mathbb{Z}[x]^m$. Moreover, one can prove that

Lemma 2.8. $\operatorname{Syz}(U)$ *is a free $\mathbb{Z}[x]$-module of rank $m - \operatorname{rank}(U)$.*

A $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$ is said to be *toric* if it is $\mathbb{Z}[x]$-saturated, that is for any nonzero $g \in \mathbb{Z}[x]$ and $\mathbf{u} \in \mathbb{Z}[x]^m$, $g\mathbf{u} \in L$ implies $\mathbf{u} \in L$.

Remark 2.9. If $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, then the syzygy module of U is obviously $\mathbb{Z}[x]$-saturated and hence toric.

For a $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$, let

$$L^C := \{\mathbf{u} \in \mathbb{Z}[x]^m \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0, \forall \mathbf{v} \in L\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\tau \mathbf{v}$ is the dot product of $\mathbf{u}$ and $\mathbf{v}$. By Lemma 2.8, L^C is a free $\mathbb{Z}[x]$-module and of rank $m - \operatorname{rank}(L)$.

Remark 2.10. For a toric $\mathbb{Z}[x]$-lattice L, one can check that $(L^C)^C = L$.

For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}[x]^m$, we denote by $\mathbb{Y}^{\mathbf{u}}$ the element $\prod_{i=1}^m y_i^{u_i}$ in $k\{y_1, \ldots, y_m\}$. $\mathbb{Y}^{\mathbf{u}}$ is called a *σ-monomial* and $\mathbf{u}$ is called its *support*.

Definition 2.11. Given a $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$, we define the binomial σ-ideal $I_L \subseteq k\{y_1, \ldots, y_m\}$ associated with L,

$$I_L := [\mathbb{Y}^{\mathbf{u}^+} - \mathbb{Y}^{\mathbf{u}^-} \mid \mathbf{u} \in L] = [\mathbb{Y}^{\mathbf{u}} - \mathbb{Y}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}[x]^m \text{ with } \mathbf{u} - \mathbf{v} \in L],$$

where $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}[x]^m$ are the positive part and the negative part of $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, respectively. L is called the *support lattice* of I_L. If L is toric, then the corresponding binomial σ-ideal I_L is called a *toric σ-ideal*.

The following two lemmas will be used later.

Lemma 2.12. *Let M be a $\mathbb{Z}[x]$-lattice. Then $M^* := \operatorname{Hom}_{\mathbb{Z}[x]}(M, \mathbb{Z}[x])$ is a free $\mathbb{Z}[x]$-module and has the same rank as M.*

Proof. Suppose $M = \mathbb{Z}[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}) := \{\sum_{i=1}^m g_i \mathbf{u}_i \mid g_i \in \mathbb{Z}[x]\} \subseteq \mathbb{Z}[x]^n$. Define a map

$$\mu \colon \mathbb{Z}[x]^m \longrightarrow M, \mathbf{e}_i \longmapsto \mathbf{u}_i,$$

where $\{\mathbf{e}_i\}_{i=1}^m$ is the standard basis of $\mathbb{Z}[x]^m$. Let $L = \ker(\mu)$. By Lemma 2.8, we have $\operatorname{rank}(L) = m - \operatorname{rank}(M)$. We define the following map

$$\alpha \colon M^* \to \mathbb{Z}[x]^m, \alpha(\varphi) = (\varphi(\mathbf{u}_1), \ldots, \varphi(\mathbf{u}_m)).$$

It is easy to see that α is an embedding and the image of α is L^C which implies $M^* \simeq L^C$. Hence M^* is free and $\operatorname{rank}(M^*) = m - \operatorname{rank}(L) = m - (m - \operatorname{rank}(M)) = \operatorname{rank}(M)$ by Lemma 2.8. $\square$

Let M be a $\mathbb{Z}[x]$-lattice. For $\mathbf{v} \in \mathbb{Z}[x]^n$, we define a $\mathbb{Z}[x]$-module homomorphism $\varphi_{\mathbf{v}}\colon M \to \mathbb{Z}[x]$ by $\varphi_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v}\rangle$, for all $\mathbf{u} \in M$. So $\varphi_{\mathbf{v}} \in M^*$ and we obtain a map

$$\theta\colon \mathbb{Z}[x]^n \to M^*, \mathbf{v} \mapsto \varphi_{\mathbf{v}}. \tag{2.1}$$

Lemma 2.13. *Let M be a $\mathbb{Z}[x]$-lattice. For any $\varphi \in M^*$, there exists a $g \in \mathbb{Z}[x]$ such that $g\varphi = \varphi_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}[x]^n$.*

Proof. The map θ in (2.1) gives an exact sequence:

$$0 \longrightarrow M^C \longrightarrow \mathbb{Z}[x]^n \xrightarrow{\theta} M^*.$$

Tensor it with $\mathbb{Q}(x)$ to obtain:

$$0 \longrightarrow M^C \otimes \mathbb{Q}(x) \longrightarrow \mathbb{Q}(x)^n \xrightarrow{\theta_{\mathbb{Q}(x)}} M^* \otimes \mathbb{Q}(x).$$

Therefore $\mathrm{rank}(\mathrm{Im}(\theta_{\mathbb{Q}(x)})) = n - \mathrm{rank}(M^C \otimes \mathbb{Q}(x)) = n - (n - \mathrm{rank}(M)) = \mathrm{rank}(M)$. It follows $\mathrm{rank}(\mathrm{Im}(\theta)) = \mathrm{rank}(M)$. Since M^* is a free $\mathbb{Z}[x]$-module of the same rank of M by Lemma 2.12, then for any $\varphi \in M^*$, there exists a $g \in \mathbb{Z}[x]$ such that $g\varphi \in \mathrm{Im}(\theta)$ as desired. □

3 Affine $P[\sigma]$-Varieties

In this section, we will introduce the concept of affine $P[\sigma]$-varieties which is a generalization of the classical σ-varieties.

3.1 Perfect $P[\sigma]$-Ideals

Firstly let us define an order on $\mathbb{Z}[x]$ as follows. We define $f = \sum_{i=0}^{p} a_i x^i > g = \sum_{i=0}^{p} b_i x^i$ if and only if there exists an integer s such that $a_i = b_i$ for $s+1 \leqslant i \leqslant p$ and $a_s > b_s$. Obviously it is a total order on $\mathbb{Z}[x]$ and $f > 0$ if and only if $\mathrm{lc}(f) > 0$ ($\mathrm{lc}(f)$ is the leading coefficient of f). Let $P[x] := \{f \in \mathbb{Z}[x] \mid f \geqslant 0\}$ and $P[x]^* := P[x]\backslash\{0\}$. We set $0^g = 0$ for $g \in P[x]^*$.

Suppose that $\mathbb{Y} = \{y_1, \ldots, y_m\}$ is a set of σ-indeterminates over k. Then the $P[\sigma]$-polynomial ring over k in $\mathbb{Y}$ is the polynomial ring in the variables y_i^g (a Laurent σ-monomial in $k\{y_i\}$), $g \in P[x]^*, i = 1, \ldots, m$. It is denoted by

$$k\{y_1, \ldots, y_m\}^{P[\sigma]}$$

and has a natural k-σ-algebra structure. An element in $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is called a $P[\sigma]$-*polynomial* over k. For $\mathbf{u} = (u_1, \ldots, u_m) \in P[x]^m$, the element $\mathbb{Y}^{\mathbf{u}} = \prod_{i=1}^{m} y_i^{u_i}$ is called a $P[\sigma]$-*monomial*. A $P[\sigma]$-*term* is the product of a constant in k and a $P[\sigma]$-monomial.

Definition 3.1. A σ-ideal $I \subseteq k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is called a $P[\sigma]$-*ideal* if $\mathbb{Y}^{\mathbf{u}} f \in I$ implies $\mathbb{Y}^{g\mathbf{u}} f \in I$ for $g \in P[x]^*$, $\mathbf{u} \in P[x]^m$, and $f \in k\{y_1, \ldots, y_m\}^{P[\sigma]}$.
A σ-ideal $I \subseteq k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is called a $P[\sigma]$-*perfect ideal* if it is a perfect σ-ideal and a $P[\sigma]$-ideal.
A σ-ideal $I \subseteq k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is called a $P[\sigma]$-*prime ideal* if it is a $P[\sigma]$-perfect ideal and a prime ideal.

Remark 3.2. For $\mathbf{u} \in P[x]^m$, we set $b_{\mathbf{u}} = (b_1, \ldots, b_m) \in \{0,1\}^m$ such that $b_i = 1$ if $u_i \neq 0$ and $b_i = 0$ if $u_i = 0$ for $i = 1, \ldots, m$. If $I \subseteq k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is a perfect σ-ideal and $\mathbb{Y}^{\mathbf{u}} f \in I$ with $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}[x]^m$, then $\mathbb{Y}^{\mathbf{u}^+} f \in I$, and therefore by the property of perfect σ-ideals, $\mathbb{Y}^{b_{\mathbf{u}}} f \in I$. Furthermore, if I is a $P[\sigma]$-perfect ideal, then $\mathbb{Y}^{\mathbf{u}} f \in I$ implies $\mathbb{Y}^{\mathbf{v}} f \in I$, for any $\mathbf{v} \in P[x]^m$ satisfying $v_i \neq 0$ if $u_i \neq 0, 1 \leqslant i \leqslant m$.

It is easy to check that the intersection of $P[\sigma]$-perfect ideals is again a $P[\sigma]$-perfect ideal. Therefore, each subset F of $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ is contained in a smallest $P[\sigma]$-perfect ideal, which is called the $P[\sigma]$-*perfect closure* of F or the $P[\sigma]$-perfect ideal generated by F. It is denoted by $\{F\}^{P[\sigma]}$.

Like the shuffling process of the perfect closure in the classical setting (see for instance [20, p.19]), we describe a shuffling process for the $P[\sigma]$-perfect closure. For a perfect σ-ideal $\mathfrak{a}$ of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$, we set

$$\mathfrak{a}' := \{\mathbb{Y}^{\mathbf{u}} f \mid \mathbf{u} \in P[x]^m, f \in k\{y_1,\ldots,y_m\}^{P[\sigma]} \text{ such that } \mathbb{Y}^{b_{\mathbf{u}}} f \in \mathfrak{a}\}. \tag{3.1}$$

Let F be a subset of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. We define $F^{[1]} = \{F\}'$ and recursively define $F^{[i]} = \{F^{[i-1]}\}'$ for $i \geqslant 2$. One can check that $\{F\}^{P[\sigma]} = \cup_{i\geqslant 1} F^{[i]}$. Moreover, we have

Lemma 3.3. *Let F and G be two subsets of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. Then*

(a) $F^{[1]}G^{[1]} \subseteq (FG)^{[1]}$;

(b) $F^{[i]}G^{[i]} \subseteq (FG)^{[i]}$ for $i \geqslant 1$;

(c) $F^{[i]} \cap G^{[i]} = (FG)^{[i]}$ for $i \geqslant 1$.

Proof. (a): Let $\mathbb{Y}^{\mathbf{u}} f \in F^{[1]}$ and $\mathbb{Y}^{\mathbf{v}} g \in G^{[1]}$. Then by (3.1), $\mathbb{Y}^{b_{\mathbf{u}}} f \in \{F\}$ and $\mathbb{Y}^{b_{\mathbf{v}}} g \in \{G\}$. So $\mathbb{Y}^{b_{\mathbf{u}}} f \cdot \mathbb{Y}^{b_{\mathbf{v}}} g \in \{F\} \cap \{G\} = \{FG\}$ ([20, Proposition 1.2.20]), and hence $\mathbb{Y}^{b_{\mathbf{u}}+b_{\mathbf{v}}} fg \in \{FG\}$. Then, we have $\mathbb{Y}^{b_{\mathbf{u}+\mathbf{v}}} fg \in \{FG\}$. It follows $\mathbb{Y}^{\mathbf{u}} f \cdot \mathbb{Y}^{\mathbf{v}} g = \mathbb{Y}^{\mathbf{u}+\mathbf{v}} fg \in (FG)^{[1]}$.
(b): We prove (b) by induction on i. The case $i = 1$ is proved by (a). For the inductive step, assume now $i \geqslant 2$. Then by (a) and the induction hypothesis,

$$\begin{aligned} F^{[i]}G^{[i]} &= (F^{[i-1]})^{[1]}(G^{[i-1]})^{[1]} \subseteq (F^{[i-1]}G^{[i-1]})^{[1]} \\ &\subseteq ((FG)^{[i-1]})^{[1]} = (FG)^{[i]}. \end{aligned}$$

(c): It is obvious that $(FG)^{[i]} \subseteq F^{[i]} \cap G^{[i]}$. For the converse, let $\mathbb{Y}^{\mathbf{u}} f \in F^{[i]} \cap G^{[i]}$, then $\mathbb{Y}^{2\mathbf{u}} f^2 \in F^{[i]}G^{[i]} \subseteq (FG)^{[i]} = \{(FG)^{[i-1]}\}'$, so by (3.1), $\mathbb{Y}^{b_{\mathbf{u}}} f^2 \in \{(FG)^{[i-1]}\}$ and hence $\mathbb{Y}^{b_{\mathbf{u}}} f \in \{(FG)^{[i-1]}\}$. It follows $\mathbb{Y}^{\mathbf{u}} f \in (FG)^{[i]}$, which proves $F^{[i]} \cap G^{[i]} \subseteq (FG)^{[i]}$. □

Proposition 3.4. Let F and G be two subsets of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. Then

$$\{F\}^{P[\sigma]} \cap \{G\}^{P[\sigma]} = \{FG\}^{P[\sigma]}.$$

Proof. Obviously, $\{F\}^{P[\sigma]} \cap \{G\}^{P[\sigma]} \supseteq \{FG\}^{P[\sigma]}$. For the converse, let $f \in \{F\}^{P[\sigma]} \cap \{G\}^{P[\sigma]}$, then there exist $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $f \in \{F\}^{[i]}$ and $f \in \{G\}^{[j]}$. Without loss of generality, we can assume $i \leqslant j$, then $f \in \{F\}^{[j]}$. Therefore, $f \in F^{[j]} \cap G^{[j]} = (FG)^{[j]} \subseteq \{FG\}^{P[\sigma]}$. □

Theorem 3.5. *Let F be a subset of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. Then $\{F\}^{P[\sigma]}$ is the intersection of all $P[\sigma]$-prime ideals containing F. In particular, every $P[\sigma]$-perfect ideal is the intersection of $P[\sigma]$-prime ideals.*

Proof. Because $P[\sigma]$-prime ideals are $P[\sigma]$-perfect, it is clear that $\{F\}^{P[\sigma]}$ is contained in every $P[\sigma]$-prime ideal containing F. It suffices to show that every $P[\sigma]$-perfect ideal is the intersection of $P[\sigma]$-prime ideals.

Let I be any $P[\sigma]$-perfect ideal of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. If f is contained in every $P[\sigma]$-prime ideal containing I, we have to show that $f \in I$. Suppose to the contrary that $f \notin I$, then we will show that there exists a $P[\sigma]$-prime ideal containing I which doesn't contain f. Let $\Sigma = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a } P[\sigma]\text{-perfect ideal}, \mathfrak{p} \supseteq I, f \notin \mathfrak{p}\}$. Clearly the union of an ascending chain of $P[\sigma]$-perfect ideals containing I not containing f is again a $P[\sigma]$-perfect ideal containing I not containing f. Since $I \in \Sigma$, Σ is a nonempty set. So by Zorn's Lemma, there exists a maximal element in Σ, denoted it by $\mathfrak{q}$. We claim that $\mathfrak{q}$ is a $P[\sigma]$-prime ideal.

We only need to show that $\mathfrak{q}$ is prime. Suppose $gh \in \mathfrak{q}$, if both g and h are not in $\mathfrak{q}$, then by the maximality of $\mathfrak{q}$, $f \in \{\mathfrak{q}, g\}^{P[\sigma]}$ and $f \in \{\mathfrak{q}, h\}^{P[\sigma]}$. So $f \in \{\mathfrak{q}, g\}^{P[\sigma]} \cap \{\mathfrak{q}, h\}^{P[\sigma]}$. By Proposition 3.4, $\{\mathfrak{q}, g\}^{P[\sigma]} \cap \{\mathfrak{q}, h\}^{P[\sigma]} = \{\mathfrak{q}^2, \mathfrak{q}g, \mathfrak{q}h, gh\}^{P[\sigma]} \subseteq \mathfrak{q}$, so $f \in \mathfrak{q}$ which is contradictory to the choice of $\mathfrak{q}$. Thus $\mathfrak{q}$ is a $P[\sigma]$-prime ideal and it contains I but doesn't contain f as desired. □

Remark 3.6. It is well known that every perfect σ-ideal is a finite intersection of σ-prime ideals, which is equivalent to the fact that any perfect σ-ideal is finitely generated as a perfect σ-ideal [20]. However, in the $P[\sigma]$-case, we still do not know whether a $P[\sigma]$-perfect ideal can be written as an intersection of finitely many $P[\sigma]$-prime ideals.

3.2 Affine $P[\sigma]$-Varieties

Let F be any subset of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$ and K any σ-field extension of k. We define the solutions of F in K to be

$$\mathbb{V}_K(F) := \{a \in K^n \mid f(a) = 0 \text{ for all } f \in F\}.$$

Definition 3.7. An *(affine) P-difference variety* or *$P[\sigma]$-variety* over k is a functor X from the category of σ-field extensions of k to the category of sets which is of the form $X = \mathbb{V}(F)$ for some subset F of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. In this situation, we say that X is the (affine) $P[\sigma]$-variety defined by F.

Obviously, the σ-affine space $\mathbb{A}_k^m = \mathbb{V}(0)$ is an affine $P[\sigma]$-variety over k. In fact, every affine σ-variety can be naturally viewed as an affine $P[\sigma]$-variety.

If X and Y are two $P[\sigma]$-varieties over k, then we write $X \subseteq Y$ to indicate that X is a subfunctor of Y. This simply means $X(K) \subseteq Y(K)$ for every $K \in \mathscr{E}_k$. In this situation, we also say that X is a *$P[\sigma]$-subvariety* of Y.

Let X be a $P[\sigma]$-subvariety of $\mathbb{A}_k^m$. Then

$$\mathbb{I}(X) := \{f \in k\{y_1,\ldots,y_m\}^{P[\sigma]} \mid f(a) = 0 \text{ for all } K \in \mathscr{E}_k \text{ and all } a \in X(K)\}$$

is called the *defining ideal* or the *vanishing ideal* of X.

Lemma 3.8. *Let X be a $P[\sigma]$-subvariety of $\mathbb{A}_k^m$. Then $\mathbb{I}(X)$ is a $P[\sigma]$-perfect ideal.*

Proof. Clearly, $\mathbb{I}(X)$ is a perfect σ-ideal. Suppose that $\mathbb{Y}^{\mathbf{u}} f \in \mathbb{I}(X)$, $\mathbf{u} \in P[x]^m$ and $g \in P[x]^*$. Then for every $K \in \mathscr{E}_k$ and for every $a \in X(K)$, $a^{\mathbf{u}} f(a) = 0$, which implies $a^{g\mathbf{u}} f(a) = 0$. It follows $\mathbb{Y}^{g\mathbf{u}} f \in \mathbb{I}(X)$. Thus $\mathbb{I}(X)$ is a $P[\sigma]$-perfect ideal. □

Definition 3.9. Let X be a $P[\sigma]$-subvariety of $\mathbb{A}_k^m$. Then the k-σ-algebra

$$k\{X\} := k\{y_1,\ldots,y_m\}^{P[\sigma]}/\mathbb{I}(X)$$

is called the $P[\sigma]$-coordinate ring of X.

A k-σ-algebra which is isomorphic to $k\{y_1,\ldots,y_m\}^{P[\sigma]}/I$ for some $P[\sigma]$-perfect ideal I is called an *affine k-$P[\sigma]$-algebra.*

Lemma 3.10. *Let X be a k-$P[\sigma]$-variety. Then for any $K \in \mathscr{E}_k$, there is a natural bijection between $X(K)$ and the set of k-σ-algebra morphisms from $k\{X\}$ to K. Indeed, $X \simeq \mathrm{Hom}(k\{X\}, -)$ as functors.*

Proof. Let $K \in \mathscr{E}_k$. Denote by $\bar{y}$ the coordinate functions on X. If $a \in X(K)$, we define a k-σ-algebra morphism from $k\{X\}$ to K by sending $\bar{y}$ to a. Conversely, given a k-σ-algebra morphism $\phi\colon k\{X\} \to K$, then $a = \phi(\bar{y})$ belongs to $X(K)$. □

Proposition 3.11. Let F be a subset of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$, then

$$\mathbb{I}(\mathbb{V}(F)) = \{F\}^{P[\sigma]}.$$

Proof. Clearly, $F \subseteq \mathbb{I}(\mathbb{V}(F))$. Since by Lemma 3.8, the ideal $\mathbb{I}(\mathbb{V}(F))$ is a $P[\sigma]$-perfect ideal, we have $\{F\}^{P[\sigma]} \subseteq \mathbb{I}(\mathbb{V}(F))$. Now suppose $f \in \mathbb{I}(\mathbb{V}(F))$, we need to show $f \in \{F\}^{P[\sigma]}$. Since $\{F\}^{P[\sigma]}$ is the intersection of all $P[\sigma]$-prime ideals containing F, it suffices to show that f lies in every $P[\sigma]$-prime ideal $\mathfrak{p}$ with $F \subseteq \mathfrak{p}$. Let K be the residue class field of $\mathfrak{p}$ and let a be the image of $\mathbb{Y} = \{y_1,\ldots,y_m\}$ in K. Since $F \subseteq \mathfrak{p}$, we find $a \in \mathbb{V}_K(F)$. And $f \in \mathbb{I}(\mathbb{V}(F))$ implies $f(a) = 0$. Therefore $f \in \mathfrak{p}$. □

Theorem 3.12. *The maps $X \mapsto \mathbb{I}(X)$ and $I \mapsto \mathbb{V}(I)$ define inclusion reversing bijections between the set of all $P[\sigma]$-subvarieties of $\mathbb{A}_k^m$ and the set of all $P[\sigma]$-perfect ideals of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$.*

Proof. Clearly, $X \subseteq \mathbb{V}(\mathbb{I}(X))$. Let $X = \mathbb{V}(F)$ be a $P[\sigma]$-subvarieties of $\mathbb{A}_k^m$. Since $F \subseteq \mathbb{I}(\mathbb{V}(F))$, we have

$$X = \mathbb{V}(F) \supseteq \mathbb{V}(\mathbb{I}(\mathbb{V}(F))) = \mathbb{V}(\mathbb{I}(X)).$$

Therefore, $\mathbb{V}(\mathbb{I}(X)) = X$.

Let I be a $P[\sigma]$-perfect ideal of $k\{y_1,\ldots,y_m\}^{P[\sigma]}$. Then by Proposition 3.11, $\mathbb{I}(\mathbb{V}(I)) = \{I\}^{P[\sigma]} = I$. □

Suppose that I is a $P[\sigma]$-ideal of $k\{y_1,\dots,y_m\}^{P[\sigma]}$. Let $k\{y_1,\dots,y_m\}^{P[\sigma]}/I$ be the σ-ring quotient of $k\{y_1,\dots,y_m\}^{P[\sigma]}$ by I and $\pi\colon k\{y_1,\dots,y_m\}^{P[\sigma]} \to k\{y_1,\dots,y_m\}^{P[\sigma]}/I$ the quotient map. Then, $P[\sigma]$-*monomials* and $P[\sigma]$-*terms* of $k\{y_1,\dots,y_m\}^{P[\sigma]}/I$ are the images of $P[\sigma]$-monomials and $P[\sigma]$-terms of $k\{y_1,\dots,y_m\}^{P[\sigma]}$ under π respectively. We define $P[\sigma]$-*perfect ideals* and $P[\sigma]$-*prime ideals* of $k\{y_1,\dots,y_m\}^{P[\sigma]}/I$ as σ-ideals of $k\{y_1,\dots,y_m\}^{P[\sigma]}/I$ whose preimages under π are $P[\sigma]$-perfect ideals and $P[\sigma]$-prime ideals of $k\{y_1,\dots,y_m\}^{P[\sigma]}$ respectively.

Corollary 3.13. Let X be an affine k-$P[\sigma]$-variety. Then the following map

$$Y \mapsto \{f \in k\{X\} \mid f(a) = 0, \forall K \in \mathscr{E}_k, \forall a \in Y(K)\}$$

is an inclusion reversing bijection between the set of all $P[\sigma]$-subvarieties of X and the set of all $P[\sigma]$-perfect ideals of $k\{X\}$.

Proof. It is clear from the definitions and Theorem 3.12. □

Definition 3.14. Let X and Y be two affine k-$P[\sigma]$-varieties. A k-σ-algebra morphism $\phi\colon k\{X\} \to k\{Y\}$ is called a *morphism of affine k-$P[\sigma]$-algebras* if ϕ maps $P[\sigma]$-terms to $P[\sigma]$-terms.

Definition 3.15. Let $X \subseteq \mathbb{A}_k^m$ and $Y \subseteq \mathbb{A}_k^n$ be two affine k-$P[\sigma]$-varieties. A morphism of functors $f\colon X \to Y$ is called a *morphism of affine k-$P[\sigma]$-varieties* if there exist $P[\sigma]$-terms $f_1,\dots,f_n \in k\{y_1,\dots,y_m\}^{P[\sigma]}$ such that $f(a) = (f_1(a),\dots,f_n(a))$ for every $K \in \mathscr{E}_k$ and every $a \in X(K)$.

Let $X \subseteq \mathbb{A}_k^m$ and $Y \subseteq \mathbb{A}_k^n$ be two affine k-$P[\sigma]$-varieties. Let $f = (f_1,\dots,f_n)\colon X \to Y$ be a morphism of affine k-$P[\sigma]$-varieties and $k\{z_1,\dots,z_n\}^{P[\sigma]}$ the $P[\sigma]$-polynomial ring in the σ-indeterminates $\{z_1,\dots,z_n\}$. We define

$$\bar{f}\colon k\{z_1,\dots,z_n\}^{P[\sigma]} \to k\{X\}, z_i \mapsto \bar{f}_i,$$

where $\bar{f}_i$ is the image of f_i in $k\{y_1,\dots,y_m\}^{P[\sigma]}/\mathbb{I}(X)$, $i = 1,\dots,n$. Then for any $K \in \mathscr{E}_k$, $a \in X(K)$ and $g \in k\{z_1,\dots,z_n\}^{P[\sigma]}$, $\bar{f}(g)(a) = g(f_1(a),\dots,f_n(a)) = g(f(a))$. Because $f(a) \in Y(K)$, we have $\bar{f}(g)(a) = 0$ if $g \in \mathbb{I}(Y)$. So $\bar{f}(g) = 0$ for $g \in \mathbb{I}(Y)$. This shows that $\bar{f}$ induces a morphism of affine k-$P[\sigma]$-algebras

$$f^*\colon k\{Y\} \to k\{X\}, \bar{z}_i \mapsto \bar{f}_i,$$

where $\bar{z}_i$ is the coordinate function on Y, $i = 1,\dots,n$. We call f^* the morphism *dual* to f, and $f^*(g) = g \circ f$, for any $g \in k\{Y\}$.

Theorem 3.16. *The category of affine k-$P[\sigma]$-varieties is antiequivalent to the category of affine k-$P[\sigma]$-algebras.*

Proof. Let $X \subseteq \mathbb{A}_k^m$ and $Y \subseteq \mathbb{A}_k^n$ be two affine k-$P[\sigma]$-varieties. We need to show that

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(k\{Y\}, k\{X\}), f \mapsto f^*$$

is bijective. First for the injectivity, let $f, g \in \mathrm{Hom}(X,Y)$ such that $f^* = g^*$. Then $h(f(a)) = f^*(h)(a) = g^*(h)(a) = h(g(a))$ for any $h \in k\{z_1,\dots,z_n\}^{P[\sigma]}$, $K \in \mathscr{E}_k$ and $a \in X(K)$. Choosing h to be the coordinate functions shows that $f = g$.

Now we show that the map is surjective. Let $\phi\colon k\{Y\} \to k\{X\}$ be a morphism of affine k-$P[\sigma]$-algebras, where $k\{Y\} = k\{z_1,\dots,z_n\}^{P[\sigma]}/\mathbb{I}(Y), k\{X\} = k\{y_1,\dots,y_m\}^{P[\sigma]}/\mathbb{I}(X)$. Suppose that $\phi(\bar{z}_i) = \bar{f}_i \in k\{X\}, i = 1,\dots,n$, where the f_i's are $P[\sigma]$-terms. Define $f\colon X \to \mathbb{A}_k^n$ by $f = (f_1,\dots,f_n)$. It is easy to check that f is actually mapping into Y. So $f\colon X \to Y$ is a morphism of k-$P[\sigma]$-varieties. Clearly $\phi = f^*$. □

Suppose that R is an affine k-$P[\sigma]$-algebra. Let $\mathrm{Spec}^{P[\sigma]}(R)$ be the set of all $P[\sigma]$-prime ideals of R, which is called the $P[\sigma]$-spectrum of R. Let $F \subseteq R$ and set

$$\mathcal{V}(F) := \{\mathfrak{p} \in \mathrm{Spec}^{P[\sigma]}(R) \mid F \subseteq \mathfrak{p}\} \subseteq \mathrm{Spec}^{P[\sigma]}(R).$$

By Theorem 3.5, we have $\mathcal{V}(F) = \mathcal{V}(\{F\}^{P[\sigma]})$. The following lemma is easy to check.

Lemma 3.17. *Let R be an affine k-$P[\sigma]$-algebra and $F, G, F_i \subseteq R$. Then*

1. $\mathcal{V}(0) = \mathrm{Spec}^{P[\sigma]}(R)$ *and* $\mathcal{V}(R) = \varnothing$;

2. $\mathcal{V}(F) \cup \mathcal{V}(G) = \mathcal{V}(FG)$;

3. $\bigcap_i \mathcal{V}(F_i) = \mathcal{V}(\bigcup_i F_i)$.

Lemma 3.17 shows that $\mathrm{Spec}^{P[\sigma]}(R)$ is a topological space with closed sets of the forms $\mathcal{V}(F), F \subseteq R$.

For $f \in R$, set $D^{P[\sigma]}(f) := \mathrm{Spec}^{P[\sigma]}(R) \backslash \mathcal{V}(f) = \{\mathfrak{p} \in \mathrm{Spec}^{P[\sigma]}(R) \mid f \notin \mathfrak{p}\}$. We call $D^{P[\sigma]}(f)$ a *basic open subset* of $\mathrm{Spec}^{P[\sigma]}(R)$.

Definition 3.18. Let X be an affine k-$P[\sigma]$-variety. Then the *topological space* of X is $\mathrm{Spec}^{P[\sigma]}(k\{X\})$ equipped with the above topology.

In analogy with Theorem 2.6, we have

Theorem 3.19. *Let X be an affine k-$P[\sigma]$-variety. There is a natural bijection between the set of equivalence classes of solutions of $\mathbb{I}(X)$ and $\mathrm{Spec}^{P[\sigma]}(k\{X\})$.*

Because of Theorem 3.19, we shall not strictly distinguish between a $P[\sigma]$-variety and its topological space. Namely, we will use X to mean the $P[\sigma]$-variety or its topological space.

4 Affine Toric $P[\sigma]$-Varieties

In this section, we will study the properties of affine toric $P[\sigma]$-varieties. Every affine toric $P[\sigma]$-variety corresponds to an affine $P[x]$-semimodule. The story is very similar to what we do for affine toric σ-varieties. But here, more properties will be proved.

4.1 Affine Toric $P[\sigma]$-Varieties and Affine $P[x]$-Semimodules

We start by recalling some basic facts about affine toric σ-varieties which were proved in [5].

Let $(\mathbb{A}^*)^n$ be the functor from $\mathscr{E}_k$ to $\mathscr{E}_k^n$ satisfying $(\mathbb{A}^*)^n(K) = (K^*)^n$ where $K \in \mathscr{E}_k$ and $K^* = K \backslash \{0\}$. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ and $\mathbb{T} = (t_1, \ldots, t_n)$ an n-tuple of σ-indeterminates. We define a functor θ^U by

$$\theta_K^U \colon (\mathbb{A}^*)^n(K) \longrightarrow \mathbb{A}^m(K), \mathbb{T} \mapsto \mathbb{T}^U = (\mathbb{T}^{\mathbf{u}_1}, \ldots, \mathbb{T}^{\mathbf{u}_m}), \tag{4.1}$$

for any $K \in \mathscr{E}_k$. The functor T_U^* from $\mathscr{E}_k$ to $\mathscr{E}_k^m$ with $T_U^*(K) = \mathrm{Im}(\theta_K^U)$ is called a *quasi σ-torus*.

Example 4.1. $(\mathbb{A}^*)^m$ is a quasi σ-torus with $U = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ the standard basis vectors of $\mathbb{Z}[x]^m$.

Definition 4.2. Given a finite set $U \subset \mathbb{Z}[x]^n$, the *affine toric σ-variety* X_U is defined to be the closure of the image of θ^U from (4.1) in $\mathbb{A}^m$ with respect to the Cohn topology.

Given a finite set $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, recall that the affine $\mathbb{N}[x]$-semimodule generated by U is $S := \mathbb{N}[x](U) = \{\sum_{i=1}^m g_i \mathbf{u}_i \mid g_i \in \mathbb{N}[x], 1 \leqslant i \leqslant m\}$. For every affine $\mathbb{N}[x]$-semimodule S, we associate it with an $\mathbb{N}[x]$-semimodule algebra $k[S]$ which is the vector space over k with S as a basis and has the multiplication induced by the addition of S. More concretely,

$$k[S] := \bigoplus_{\mathbf{u} \in S} k\chi^{\mathbf{u}} = \{\sum_{\mathbf{u} \in S} c_{\mathbf{u}} \chi^{\mathbf{u}} \mid c_{\mathbf{u}} \in k \text{ and } c_{\mathbf{u}} = 0 \text{ for all but finitely many } \mathbf{u}\}$$

with the multiplication induced by $\chi^{\mathbf{u}} \cdot \chi^{\mathbf{v}} = \chi^{\mathbf{u}+\mathbf{v}}$, for $\mathbf{u}, \mathbf{v} \in S$. Make $k[S]$ to be a k-σ-algebra by defining $\sigma(\chi^{\mathbf{u}}) = \chi^{x\mathbf{u}}$, for $\mathbf{u} \in S$.

In [5, Theorem 3.5], it was proved that an affine σ-variety X is toric if and only if there exists an affine $\mathbb{N}[x]$-semimodule S such that $X \simeq \mathrm{Spec}^{\sigma}(k[S])$. Furthermore, the category of affine toric σ-varieties with toric morphisms is antiequivalent to the category of affine $\mathbb{N}[x]$-semimodules with $\mathbb{N}[x]$-semimodule morphisms ([5, Theorem 6.5]).

Each affine toric σ-variety contains a σ-torus as an open subset whose group action extends to the affine toric σ-variety. In [5, Theorem 5.5], it was proved that an affine σ-variety T is a σ-torus if and only if there exists a $\mathbb{Z}[x]$-lattice M such that $T \simeq \mathrm{Spec}^{\sigma}(k[M])$.

Remark 4.3. Let $T = \mathrm{Spec}^{\sigma}(k[M])$ be a σ-torus. A *character* of T is a morphism of σ-algebraic groups $\chi\colon T \to (\mathbb{A}^*)^1$. Denote all characters of T by $X(T)$. Then $X(T) \subseteq k[M]$. Every $\mathbf{u} \in M$ gives a character $\chi^{\mathbf{u}}\colon T \to (\mathbb{A}^*)^1$ which satisfies that for each $K \in \mathscr{E}_k$ and an element ϕ of $T(K)$, $\chi^{\mathbf{u}}(\phi) = \phi(\mathbf{u}) \in K^*$. Actually, all characters of T arise in this way. Thus $X(T) \simeq M$. In particular, $X((\mathbb{A}^*)^n) \simeq \mathbb{Z}[x]^n$.

Now we give the definition of affine toric $P[\sigma]$-varieties.

Definition 4.4. Given a finite set $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, the *affine toric $P[\sigma]$-variety* $\mathscr{X}_U$ is the closure of the quasi σ-torus $T_U^* \subseteq \mathbb{A}^m$ in $\mathbb{A}^m$ under the topology in Definition 3.18. More precisely, let

$$J_U := \{f \in k\{y_1, \ldots, y_m\}^{P[\sigma]} \mid f(\mathbb{T}^{\mathbf{u}_1}, \ldots, \mathbb{T}^{\mathbf{u}_m}) = 0\}. \tag{4.2}$$

Then the affine toric $P[\sigma]$-variety defined by U is $\mathscr{X}_U = \mathbb{V}(J_U)$.

Example 4.5. $\mathbb{A}^m$ is an affine toric $P[\sigma]$-variety with quasi σ-torus $(\mathbb{A}^*)^m$.

For $\mathbf{v} = (v_i) \in \mathbb{Z}[x]^m$, let $I = \{i \mid v_i \in P[x]\}$. We write $\mathbf{v}_+ \in P[x]^m$ with $(\mathbf{v}_+)_i = v_i$ for $i \in I$, and $(\mathbf{v}_+)_i = 0$ for $i \in \{1, \ldots, m\} \backslash I$; and write $\mathbf{v}_- \in P[x]^m$ with $(\mathbf{v}_-)_i = -v_i$ for $i \in \{1, \ldots, m\} \backslash I$, and $(\mathbf{v}_-)_i = 0$ for $i \in I$. Then $\mathbf{v} = \mathbf{v}_+ - \mathbf{v}_-$.

Definition 4.6. Given a $\mathbb{Z}[x]$-lattice $L \subseteq \mathbb{Z}[x]^m$, we define a binomial $P[\sigma]$-ideal $J_L \subseteq k\{y_1, \ldots, y_m\}^{P[\sigma]}$ associated with L

$$J_L := [\mathbb{Y}^{\mathbf{v}_1} - \mathbb{Y}^{\mathbf{v}_2} \mid \mathbf{v}_1 - \mathbf{v}_2 \in L, \mathbf{v}_1, \mathbf{v}_2 \in P[x]^m] = [\mathbb{Y}^{\mathbf{v}_+} - \mathbb{Y}^{\mathbf{v}_-} \mid \mathbf{v} \in L].$$

L is called the *support lattice* of J_L. If L is a toric $\mathbb{Z}[x]$-lattice, then J_L is called a *toric $P[\sigma]$-ideal*.

Lemma 4.7. *Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ and $\mathscr{X}_U$ the affine toric $P[\sigma]$-variety defined by U. Then $J_U = \mathbb{I}(\mathscr{X}_U)$ is a toric $P[\sigma]$-ideal whose support lattice is $L = \mathrm{Syz}(U)$.*

Proof. By Remark 2.9, L is a toric $\mathbb{Z}[x]$-lattice. Then it suffices to show that $J_U = J_L$, where J_U is defined in (4.2). For $\mathbf{v} \in L$, we have $(\mathbb{Y}^{\mathbf{v}} - 1)(\mathbb{T}^U) = (\mathbb{T}^U)^{\mathbf{v}} - 1 = \mathbb{T}^{U\mathbf{v}} - 1 = 0$, where we regard U as the matrix with the columns $\mathbf{u}_i, i = 1, \ldots, m$. As a consequence, $(\mathbb{Y}^{\mathbf{v}_+} - \mathbb{Y}^{\mathbf{v}_-})(\mathbb{T}^U) = 0$ and $\mathbb{Y}^{\mathbf{v}_+} - \mathbb{Y}^{\mathbf{v}_-} \in J_U$ with $\mathbf{v} \in L$. Since J_L is generated by $\mathbb{Y}^{\mathbf{v}_+} - \mathbb{Y}^{\mathbf{v}_-}$ for $\mathbf{v} \in L$, we have $J_L \subseteq J_U$.

To prove the other direction of the inclusion, let us consider the following map

$$\theta\colon k\{y_1, \ldots, y_m\}^{P[\sigma]} \to k\{t_1^{\pm 1}, \ldots, t_n^{\pm 1}\}, f \mapsto f(\mathbb{T}^{\mathbf{u}_1}, \ldots, \mathbb{T}^{\mathbf{u}_m}).$$

Define a grading on $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ by $\deg(\mathbb{Y}^{\mathbf{v}}) = U\mathbf{v} \in \mathbb{Z}[x]^n$, where we regard U as the matrix with the columns $\mathbf{u}_i, i = 1, \ldots, m$. The grading on $k\{t_1^{\pm 1}, \ldots, t_n^{\pm 1}\}$ is given by $\deg(\mathbb{T}^{\mathbf{u}}) = \mathbf{u} \in \mathbb{Z}[x]^n$. Then $k\{y_1, \ldots, y_m\}^{P[\sigma]}$ and $k\{t_1^{\pm 1}, \ldots, t_n^{\pm 1}\}$ are both $\mathbb{Z}[x]^n$-graded. It is easy to check that θ is a homogeneous map of degree $\mathbf{0}$. It follows that the kernel of θ is homogeneous. So an element of $\ker(\theta)$ of degree $\mathbf{u}$ can be written as $\sum_{U\mathbf{v}=\mathbf{u}} \alpha_{\mathbf{v}} \mathbb{Y}^{\mathbf{v}}$ with $\sum_{\mathbf{v}} \alpha_{\mathbf{v}} = 0$. Such an element is in J_L. Hence $J_U = \ker(\theta) \subseteq J_L$. □

The following lemma shows that the converse of Lemma 4.7 is also valid.

Lemma 4.8. *If I is a toric $P[\sigma]$-ideal in $k\{y_1, \ldots, y_m\}^{P[\sigma]}$, then $\mathbb{V}(I)$ is an affine toric $P[\sigma]$-variety.*

Proof. Since I is a toric $P[\sigma]$-ideal, then the $\mathbb{Z}[x]$-lattice corresponding to I, denoted by L, is toric. Suppose that $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{Z}[x]^m$ is a set of generators of L^C. Regard V as a matrix with columns $\mathbf{v}_i, i = 1, \ldots, n$ and let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ be the set of the row vectors of V. Let us consider the affine toric $P[\sigma]$-variety $\mathscr{X}_U$ defined by U. To prove the lemma, it suffices to show $\mathscr{X}_U = \mathbb{V}(I)$ or $J_U = I$. Since toric $P[\sigma]$-ideals and toric $\mathbb{Z}[x]$-lattices are in a one-to-one correspondence and the support lattice of J_U is $\mathrm{Syz}(U)$ by Lemma 4.7, we only need to show $\mathrm{Syz}(U) = L$. This is clear since $\mathrm{Syz}(U) = \ker(V^\tau) = (L^C)^C = L$ by Remark 2.10. □

Combining Lemma 4.7 and Lemma 4.8, we have

Theorem 4.9. *An affine $P[\sigma]$-variety X is toric if and only if $\mathbb{I}(X)$ is a toric $P[\sigma]$-ideal.*

Now we introduce the concept of affine $P[x]$-semimodules.

A subset $S \subseteq \mathbb{Z}[x]^n$ is called a *$P[x]$-semimodule* if it satisfies (i) if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$, and (ii) if $g \in P[x]$ and $\mathbf{u} \in S$, then $g\mathbf{u} \in S$. Moreover, if there exists a finite subset $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$

such that $S = P[x](U) := \{\sum_{i=1}^m g_i\mathbf{u}_i \mid g_i \in P[x], 1 \leqslant i \leqslant m\}$, then S is called an *affine $P[x]$-semimodule*. If S is an affine $P[x]$-semimodule, let $S^{md} := \{\mathbf{u} - \mathbf{v} \mid \mathbf{u}, \mathbf{v} \in S\}$ be the $\mathbb{Z}[x]$-lattice generated by S, and we define rank(S) as the rank of the $\mathbb{Z}[x]$-lattice S^{md}. A map $\phi\colon S \to S'$ between two $P[x]$-semimodules is a *$P[x]$-semimodule morphism* if ϕ satisfies $\phi(\mathbf{u}+\mathbf{v}) = \phi(\mathbf{u}) + \phi(\mathbf{v}), \phi(g\mathbf{u}) = g\phi(\mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in S$ and $g \in P[x]$.

Remark 4.10. For an affine $P[x]$-semimodule $S = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\})$, if some $\mathbf{u}_i$ can be generated by the other $\{\mathbf{u}_j\}_{j\neq i}$, i.e. there exist $g_j \in P[x], j \neq i$ such that $\mathbf{u}_i = \sum_{j\neq i} g_j\mathbf{u}_j$, then we can delete $\mathbf{u}_i$ from $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ to generate the same affine $P[x]$-semimodule. We say that a set of generators $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of S is reduced if there is no such $\mathbf{u}_i$ in $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and we always assume that the set of generators of an affine $P[x]$-semimodule S is reduced in the following.

For every affine $P[x]$-semimodule S, we define a *$P[x]$-semimodule algebra* $k[S]$ which is the vector space over k with S as a basis and the multiplication induced by the addition of S. More concretely,

$$k[S] := \bigoplus_{\mathbf{u}\in S} k\chi^{\mathbf{u}} = \{\sum_{\mathbf{u}\in S} c_{\mathbf{u}}\chi^{\mathbf{u}} \mid c_{\mathbf{u}} \in k \text{ and } c_{\mathbf{u}} = 0 \text{ for all but finitely many } \mathbf{u}\},$$

with the multiplication induced by

$$\chi^{\mathbf{u}_1} \cdot \chi^{\mathbf{u}_2} = \chi^{\mathbf{u}_1+\mathbf{u}_2}.$$

Make $k[S]$ to be a k-σ-algebra by defining

$$\sigma(\chi^{\mathbf{u}}) = \chi^{x\mathbf{u}}, \text{ for } \mathbf{u} \in S.$$

Suppose $S = P[x](U) = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\})$, then $k[S] = k\{\chi^{\mathbf{u}_1}, \ldots, \chi^{\mathbf{u}_m}\}^{P[\sigma]}$. When an embedding $S \to \mathbb{Z}[x]^n$ is given, it induces an embedding $k[S] \to k[\mathbb{Z}[x]^n] \simeq k\{t_1^{\pm1}, \ldots, t_n^{\pm1}\}$ via $\chi^{\mathbf{u}}$ mapping to $\mathbb{T}^{\mathbf{u}}$, where $\mathbb{T} = \{t_1, \ldots, t_n\}$ is a set of σ-indeterminates. Therefore, $k[S]$ is a k-σ-subalgebra of $k\{t_1^{\pm1}, \ldots, t_n^{\pm1}\}$ and it follows that $k[S]$ is a σ-domain. Also, we can view $k[S]$ as an S-graded ring by defining $\deg(\chi^{\mathbf{u}}) = \mathbf{u}$. We will see that $k[S]$ is actually the $P[\sigma]$-coordinate ring of an affine toric $P[\sigma]$-variety.

Theorem 4.11. *Let X be an affine k-$P[\sigma]$-variety. Then X is toric if and only if there exists an affine $P[x]$-semimodule S such that $X \simeq \mathrm{Spec}^{P[\sigma]}(k[S])$. Equivalently, the $P[\sigma]$-coordinate ring of X is $k[S]$.*

Proof. Suppose that $X = \mathscr{X}_U$ is the affine toric $P[\sigma]$-variety defined by $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ and J_U is defined in (4.2). Let $S = P[x](U)$ be the affine $P[x]$-semimodule generated by U. Define the following k-σ-algebra homomorphism

$$\theta\colon k\{y_1, \ldots, y_m\}^{P[\sigma]} \longrightarrow k[S], y_i \mapsto \chi^{\mathbf{u}_i}, i = 1, \ldots, m.$$

The map θ is surjective by the definition of $k[S]$. If $f \in \ker(\theta)$, then $f(\chi^{\mathbf{u}_1}, \ldots, \chi^{\mathbf{u}_m}) = 0$, which is equivalent to $f \in J_U$. It follows that $\ker(\theta) = J_U$ and $k\{y_1, \ldots, y_m\}^{P[\sigma]}/J_U \simeq k[S]$. Therefore, $X \simeq \mathrm{Spec}^{P[\sigma]}(k\{y_1, \ldots, y_m\}^{P[\sigma]}/J_U) = \mathrm{Spec}^{P[\sigma]}(k[S])$.

Conversely, if $X \simeq \mathrm{Spec}^{P[\sigma]}(k[S])$, where $S \subseteq \mathbb{Z}[x]^n$ is an affine $P[x]$-semimodule, and assume that $S = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\})$. Let $\mathscr{X}_U$ be the affine toric $P[\sigma]$-variety defined by $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. Then as we just proved, the $P[\sigma]$-coordinate ring of $\mathscr{X}_U$ is isomorphic to $k[S]$. Then $X \simeq \mathscr{X}_U$. □

Suppose that S is an affine $P[x]$-semimodule. For each $K \in \mathscr{E}_k$, a map $\phi\colon S \to K$ is a *morphism* from S to K if ϕ satisfies $\phi(\sum_i g_i\mathbf{u}_i) = \prod_i \phi(\mathbf{u}_i)^{g_i}$, for $\mathbf{u}_i \in S$ and $g_i \in P[x]$.

Corollary 4.12. Let $X = \mathrm{Spec}^{P[\sigma]}(k[S])$ be an affine toric $P[\sigma]$-variety. Then for each $K \in \mathscr{E}_k$, there is a one-to-one correspondence between $X(K)$ and $\mathrm{Hom}(S, K)$. Equivalently, $X \simeq \mathrm{Hom}(S, -)$ as functors.

Proof. By Lemma 3.10, for each $K \in \mathscr{E}_k$, an element of $X(K)$ is given by a k-σ-algebra homomorphism $f\colon k[S] \to K$. Then f induces a morphism $\bar{f}\colon S \to K$ such that $\bar{f}(\mathbf{u}) = f(\chi^{\mathbf{u}})$ for $\mathbf{u} \in S$. Conversely, given a morphism $\varphi\colon S \to K$, φ extends to a k-σ-algebra homomorphism $\varphi^*\colon k[S] \to K$ which proves the one-to-one correspondence. □

In the rest of this paper, we always identity an element of $X(K)$ with a morphism from S to K for each $K \in \mathscr{E}_k$.

We have the following definition for $P[\sigma]$-tori.

Definition 4.13. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$. The $P[\sigma]$-*torus* $\widetilde{T_U}$ defined by U is the closure of the quasi σ-torus $T_U^* \subseteq \mathbb{A}^m$ in $(\mathbb{A}^*)^m$ under the topology in Definition 3.18.

Remark 4.14. The above definition can be stated in a more precise way. Note that $(\mathbb{A}^*)^m$ is isomorphic to the affine $P[\sigma]$-variety defined by $I_0 = [y_1z_1 - 1, \ldots, y_mz_m - 1]$ in $(\mathbb{A})^{2m}$. Furthermore, the map

$$\theta\colon (\mathbb{A}^*)^m \longrightarrow (\mathbb{A})^{2m} \tag{4.3}$$

defined by $\theta(a_1, \ldots, a_m) = (a_1, \ldots, a_m, a_1^{-1}, \ldots, a_m^{-1})$ gives a one-to-one correspondence between $P[\sigma]$-subvarieties of $(\mathbb{A}^*)^m$ and affine $P[\sigma]$-varieties contained in $\mathbb{V}(I_0)$. Then the $P[\sigma]$-torus $\widetilde{T_U}$ is the preimage of the affine $P[\sigma]$-variety $\mathscr{X}_{U\cup(-U)}$ in $\mathbb{A}^{2m}$ under the map θ, where $-U = \{-\mathbf{u}_1, \ldots, -\mathbf{u}_m\}$.

Suppose $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$. Let $\widetilde{T_U}$ and $\mathscr{X}_U$ be the $P[\sigma]$-torus and the affine toric $P[\sigma]$-variety defined by U respectively. Then by definition, $\widetilde{T_U} = \mathscr{X}_U \cap (\mathbb{A}^*)^m$.

Proposition 4.15. Let $\widetilde{T}$ be an affine $P[\sigma]$-variety. Then $\widetilde{T}$ is a $P[\sigma]$-torus if and only if there exists a $\mathbb{Z}[x]$-lattice M such that $\widetilde{T} \simeq \mathrm{Spec}^{P[\sigma]}(k[M])$.

Proof. Suppose that $\widetilde{T}$ is defined by U and let $M = \mathbb{Z}[x](U)$. Since $\widetilde{T} \simeq \mathscr{X}_{U\cup(-U)}$ (Remark 4.14), we just need to show that the $P[\sigma]$-coordinate ring of $\mathscr{X}_{U\cup(-U)}$ is $k[M]$. By definition, $\mathscr{X}_{U\cup(-U)}$ is the affine toric $P[\sigma]$-variety defined by $U \cup (-U)$. Thus by Theorem 4.11, the $P[\sigma]$-coordinate ring of $\mathscr{X}_{U\cup(-U)}$ is $k[P[x](U \cup (-U))] = k[M]$.

Conversely, suppose that U is a finite subset of $\mathbb{Z}[x]^n$ and $M = \mathbb{Z}[x](U)$. Then by the proof of the above necessity, U defines a $P[\sigma]$-torus $\widetilde{T_U}$ whose $P[\sigma]$-coordinate ring is $k[M]$. Since $\widetilde{T} \simeq \widetilde{T_U}$, $\widetilde{T}$ is a $P[\sigma]$-torus. □

As a corollary of Proposition 4.15, we have

Corollary 4.16. Suppose that S is an affine $P[x]$-semimodule. Let $X = \mathrm{Spec}^{P[\sigma]}(k[S])$ and $\widetilde{T} = \mathrm{Spec}^{P[\sigma]}(k[S^{md}])$. For each $K \in \mathscr{E}_k$, there is a one-to-one correspondence between elements of $\widetilde{T}(K)$ and $\mathrm{Hom}(S^{md}, K)$. Equivalently, $\widetilde{T} \simeq \mathrm{Hom}(S^{md}, -)$ as functors.

Proof. For each $K \in \mathscr{E}_k$, suppose that $\gamma\colon S \to K$ is an element of $X(K)$ which lies in $\widetilde{T}(K)$. Since elements of $\widetilde{T}(K)$ are invertible, $\gamma(S) \subseteq K^*$ and hence γ can be extended to S^{md} by defining $\gamma(-\mathbf{u}) = 1/\gamma(\mathbf{u}), \mathbf{u} \in S$. So the one-to-one correspondence follows from Proposition 4.12. □

Let M be a $\mathbb{Z}[x]$-lattice. Let $T = \mathrm{Spec}^{\sigma}(k[M])$ and $\widetilde{T} = \mathrm{Spec}^{P[\sigma]}(k[M])$ be the σ-torus and the $P[\sigma]$-torus associated with M respectively. Since by [5, Proposition 5.7], $T \simeq \mathrm{Hom}(M, -)$ and by Corollary 4.16, $\widetilde{T} \simeq \mathrm{Hom}(M, -)$, we know that $T = \widetilde{T}$ as functors. So a $P[\sigma]$-torus is actually a σ-torus and we will write T for $\widetilde{T}$.

Proposition 4.17. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ and $\mathscr{X}_U$ the affine toric $P[\sigma]$-variety defined by U. Then $\mathscr{X}_U$ is an irreducible $P[\sigma]$-variety containing the σ-torus T_U as an open subset and is of σ-dimension $\mathrm{rank}(U)$.

Proof. Let $S = P[x](U)$. Since $\mathscr{X}_U$ has the $P[\sigma]$-coordinate ring $k[S]$ by Theorem 4.11 and $k[S]$ is a σ-domain, then $\mathscr{X}_U$ is irreducible. The inclusion $i\colon S \hookrightarrow M = S^{md}$ induces a k-σ-algebra homomorphism $i^*\colon k[S] \hookrightarrow k[M]$ which corresponds to a morphism $j\colon T_U \to \mathscr{X}_U$. S^{md} is generated as a $P[x]$-semimodule by S and $-(\mathbf{u}_1 + \ldots + \mathbf{u}_m)$. This implies that $k[M]$ is the localization of $k[S]$ at the element $\chi^{\mathbf{u}_1+\ldots+\mathbf{u}_m}$. Therefore, j embeds T_U into $\mathscr{X}_U$ as a principal affine open subset. Since the σ-dimension of T_U is $\mathrm{rank}(U)$, the σ-dimension of $\mathscr{X}_U$ is equal to $\mathrm{rank}(U)$. □

4.2 σ-Algebraic Group Action on X

A σ-variety G is called a *σ-algebraic group* if G has a group structure and the group multiplication and the inverse map are both morphisms of σ-varieties (see [21]).

Definition 4.18. Let G be a σ-algebraic group and X a $P[\sigma]$-variety. We say that G has a *σ-algebraic group action* on X if there exists a morphism of $P[\sigma]$-varieties (regrading G as a $P[\sigma]$-variety)

$$\phi\colon G \times X \longrightarrow X$$

such that for any $K \in \mathscr{E}_k$,

$$\phi_K \colon G(K) \times X(K) \longrightarrow X(K)$$

is a group action of $G(K)$ on $X(K)$.

As in the σ-case, there is a σ-algebraic group action on an affine toric $P[\sigma]$-variety. Given an affine $P[x]$-semimodule S, let X and T be the affine toric $P[\sigma]$-variety and the σ-torus associated with S respectively. We describe how T acts on X as a σ-algebraic group. Define a map $X \times X \to X \colon ((x_1, \ldots, x_m), (y_1, \ldots, y_m)) \mapsto (x_1y_1, \ldots, x_my_m)$, for $(x_1, \ldots, x_m), (y_1, \ldots, y_m) \in X$. It can be described using $P[x]$-semimodule morphisms as follows: for any $K \in \mathscr{E}_k$, let $\varphi, \psi \colon S \to K$ be two elements of $X(K)$, then $(\varphi, \psi) \mapsto \varphi\psi \colon S \to K, \varphi\psi(\mathbf{u}) = \varphi(\mathbf{u})\psi(\mathbf{u})$, for $\mathbf{u} \in S$. This corresponds to the k-σ-algebra homomorphism $\Phi \colon k[S] \to k[S] \otimes k[S]$ such that $\Phi(\chi^{\mathbf{u}}) = \chi^{\mathbf{u}} \otimes \chi^{\mathbf{u}}$, for $\mathbf{u} \in S$.

Via the embedding $T \subseteq X$, the operation on X induces a map $T \times X \to X$ which is clearly a σ-algebraic group action on X and extends the group action of T on itself. It corresponds to the k-σ-algebra homomorphism $k[S] \to k[S^{md}] \otimes k[S]$, $\chi^{\mathbf{u}} \mapsto \chi^{\mathbf{u}} \otimes \chi^{\mathbf{u}}$, for $\mathbf{u} \in S$.

The following theorem shows that if an affine $P[\sigma]$-variety contains a σ-torus as an open subset extending the group action of the σ-torus on itself, then it is toric. In other words, the theorem gives a description of affine toric $P[\sigma]$-varieties in terms of σ-algebraic group actions.

Theorem 4.19. *Let X be an affine k-$P[\sigma]$-variety, $T \subseteq X$ an open subset which is a σ-torus such that the group action of T on itself extends to a σ-algebraic group action on X. Then there is an affine $P[x]$-semimodule S and an isomorphism $X \simeq \mathrm{Spec}^{P[\sigma]}(k[S])$. In other words, X is an affine toric $P[\sigma]$-variety.*

Proof. By Proposition 4.15, there is a $\mathbb{Z}[x]$-lattice M such that $T \simeq \mathrm{Spec}^{P[\sigma]}(k[M])$. The open embedding $T \subseteq X$ induces $k\{X\} \subseteq k[M]$. Since the action of T on itself extends to a σ-algebraic group action on X, we have the following commutative diagram:

$$\begin{tikzcd} T \times T \arrow[r, "\phi"] \arrow[d] & T \arrow[d] \\ T \times X \arrow[r, "\tilde{\phi}"] & X \end{tikzcd} \tag{4.4}$$

where ϕ is the group action of T, $\tilde{\phi}$ is the extension of ϕ to $T \times X$.

From (4.4), we obtain the following commutative diagram of the corresponding $P[\sigma]$-coordinate rings:

$$\begin{tikzcd} k\{X\} \arrow[r, "\tilde{\Phi}"] \arrow[d] & k[M] \otimes_k k\{X\} \arrow[d] \\ k[M] \arrow[r, "\Phi"] & k[M] \otimes_k k[M] \end{tikzcd}$$

where the vertical maps are inclusions, and $\Phi(\chi^{\mathbf{u}}) = \chi^{\mathbf{u}} \otimes \chi^{\mathbf{u}}$ for $\mathbf{u} \in M$. It follows that if $\sum_{\mathbf{u} \in M} \alpha_{\mathbf{u}}\chi^{\mathbf{u}}$ with finitely many $\alpha_{\mathbf{u}} \neq 0$ is in $k\{X\}$, then $\sum_{\mathbf{u} \in M} \alpha_{\mathbf{u}}\chi^{\mathbf{u}} \otimes \chi^{\mathbf{u}}$ is in $k[M] \otimes_k k\{X\}$. So $\alpha_{\mathbf{u}}\chi^{\mathbf{u}} \in k\{X\}$ for every $\mathbf{u} \in M$. This shows that there is a subset S of M such that $k\{X\} = k[S] = \bigoplus_{\mathbf{u} \in S} k\chi^{\mathbf{u}}$. Since $k\{X\}$ is an affine k-$P[\sigma]$-subalgebra of $k[M]$, it follows that S is an affine $P[x]$-semimodule. So by Theorem 4.11, X is an affine toric $P[\sigma]$-variety. □

4.3 Toric Morphisms between Affine Toric $P[\sigma]$-Varieties

Note that if $\phi \colon S_1 \to S_2$ is a morphism between affine $P[x]$-semimodules, we have an induced k-σ-algebra homomorphism $\bar{\phi} \colon k[S_1] \to k[S_2]$ such that $\bar{\phi}(\chi^{\mathbf{u}}) = \chi^{\phi(\mathbf{u})}$, for all $\mathbf{u} \in S_1$, which gives a morphism between affine toric $P[\sigma]$-varieties $\phi^* \colon \mathrm{Spec}^{P[\sigma]}(k[S_2]) \to \mathrm{Spec}^{P[\sigma]}(k[S_1])$. In this subsection, we will show that actually all toric morphisms between affine toric $P[\sigma]$-varieties arise in this way. First we give the definition of toric morphisms.

Definition 4.20. Let $X_i = \mathrm{Spec}^{P[\sigma]}(k[S_i])$ be the affine toric $P[\sigma]$-varieties coming from affine $P[x]$-semimodules S_i with σ-tori T_i, $i = 1, 2$ respectively. A morphism $\phi\colon X_1 \to X_2$ is said to be *toric* if $\phi(T_1) \subseteq T_2$ and $\phi|_{T_1}$ is a σ-algebraic group homomorphism.

Proposition 4.21. Let $\phi\colon X_1 \to X_2$ be a toric morphism of affine toric $P[\sigma]$-varieties. Then ϕ preserves group actions, namely,

$$\phi(t \cdot p) = \phi(t) \cdot \phi(p),$$

for all $t \in T_1$ and $p \in X_1$.

Proof. Suppose that the action of T_i on X_i is given by the morphism $\varphi_i\colon T_i \times X_i \to X_i, i = 1, 2$. To show ϕ preserves group actions is equivalent to showing that the following diagram is commutative:

$$\begin{tikzcd} T_1 \times X_1 \arrow[r, "\varphi_1"] \arrow[d, "\phi|_{T_1} \times \phi"'] & X_1 \arrow[d, "\phi"] \\ T_2 \times X_2 \arrow[r, "\varphi_2"] & X_2 \end{tikzcd} \tag{4.5}$$

If we replace X_i by T_i in the diagram, $i = 1, 2$, then it certainly commutes since $\phi|_{T_1}$ is a σ-algebraic group homomorphism. And since $T_1 \times T_1$ is dense in $T_1 \times X_1$, the whole diagram is commutative. □

The following lemma is taken from [5, Lemma 6.3].

Lemma 4.22. *Let T_i be the σ-tori associated with the $\mathbb{Z}[x]$-lattices $M_i, i = 1, 2$ respectively. Then a map $\phi\colon T_1 \to T_2$ is a σ-algebraic group homomorphism if and only if the corresponding map of the σ-coordinate rings $\phi^*\colon k[M_2] \to k[M_1]$ is induced by a $\mathbb{Z}[x]$-module homomorphism $\hat{\phi}\colon M_2 \to M_1$.*

Theorem 4.23. *Let $X_i = \mathrm{Spec}^{P[\sigma]}(k[S_i])$ be affine toric $P[\sigma]$-varieties coming from affine $P[x]$-semimodules S_i with σ-tori $T_i, i = 1, 2$ respectively. Then a morphism $\phi\colon X_1 \to X_2$ is toric if and only if it is induced by a $P[x]$-semimodule morphism $\hat{\phi}\colon S_2 \to S_1$.*

Proof. For the sufficiency, suppose that $\hat{\phi}\colon S_2 \to S_1$ is a $P[x]$-semimodule morphism. Then $\hat{\phi}$ extends to a $\mathbb{Z}[x]$-module homomorphism $\hat{\phi}\colon M_2 \to M_1$, where $M_1 = S_1^{md}, M_2 = S_2^{md}$. By Lemma 4.22, it induces a morphism of σ-algebraic groups $\phi\colon T_1 \to T_2$. So ϕ is toric.

For the necessity, note that ϕ induces $\phi^*\colon k[S_2] \to k[S_1]$. Since ϕ is toric, $\phi|_{T_1}$ is a σ-algebraic group homomorphism. By Lemma 4.22, it is induced by a $\mathbb{Z}[x]$-module homomorphism $\tilde{\phi}\colon M_2 \to M_1$. This, combined with $\phi^*(k[S_2]) \subseteq k[S_1]$, implies that $\tilde{\phi}$ induces a $P[x]$-semimodule morphism $\hat{\phi}\colon S_2 \to S_1$. □

Combining Theorem 4.11 with Theorem 4.23, we have

Theorem 4.24. *The category of affine toric $P[\sigma]$-varieties with toric morphisms is antiequivalent to the category of affine $P[x]$-semimodules with $P[x]$-semimodule morphisms.*

4.4 T-Orbits of Affine Toric $P[\sigma]$-Varieties

In this subsection, we will establish a one-to-one correspondence between the irreducible T-invariant $P[\sigma]$-subvarieties of an affine toric $P[\sigma]$-variety and the faces of the corresponding affine $P[x]$-semimodule. Also, a one-to-one correspondence between the T-orbits of an affine toric $P[\sigma]$-variety and the faces of corresponding affine $P[x]$-semimodules is given for a class of affine $P[x]$-semimodules.

Definition 4.25. Let S be an affine $P[x]$-semimodule. Define a *face* of S to be a $P[x]$-subsemimodule $F \subseteq S$ such that

1. for $\mathbf{u}_1, \mathbf{u}_2 \in S$, $\mathbf{u}_1 + \mathbf{u}_2 \in F$ implies $\mathbf{u}_1, \mathbf{u}_2 \in F$;
2. for $g \in P[x]^*$ and $\mathbf{u} \in S$, $g\mathbf{u} \in F$ implies $\mathbf{u} \in F$.

If F is a face of S, then we write $F \preceq S$.

Note that if $S = P[x](\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\})$, and F is a face of S, then F is generated by a subset of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ as a $P[x]$-semimodule. It follows that F is an affine $P[x]$-semimodule and S has only finitely many faces. S is a face of itself. A *proper face* of S is a face strictly contained in S. It is easy

to check that the intersection of two faces is again a face and a face of a face is again a face. A face of rank one is called an *edge*, and a face of rank $\mathrm{rank}(S)-1$ is called a *facet*. Note that unlike affine $\mathbb{N}[x]$-semimodules, a proper face of an affine $P[x]$-semimodule S must have rank less than $\mathrm{rank}(S)$. S is said to be *pointed* if $S\cap(-S)=\{\mathbf{0}\}$, i.e. $\{\mathbf{0}\}$ is a face of S.

Example 4.26. Let $S=P[x](\{x-1,x-2\})$. Then S has two faces: $F_1=\{0\}$ and $F_2=S$.

Example 4.27. Let $S=P[x](\{\mathbf{u}_1=(x,1),\mathbf{u}_2=(x,2),\mathbf{u}_3=(x,3)\})$. Then S has four faces: $F_1=\{\mathbf{0}\}$, $F_2=P[x](\mathbf{u}_1)$, $F_3=P[x](\mathbf{u}_3)$ and $F_4=S$.

Example 4.28. Let $S=P[x](\{\mathbf{u}_1=(x,1,1),\mathbf{u}_2=(1,x,1),\mathbf{u}_3=(1,1,x),\mathbf{u}_4=(1,1,1)\})$. Then S has eight faces: $F_1=\{\mathbf{0}\}$, $F_2=P[x](\mathbf{u}_1)$, $F_3=P[x](\mathbf{u}_2)$, $F_4=P[x](\mathbf{u}_3)$, $F_5=P[x](\{\mathbf{u}_2,\mathbf{u}_3\})$, $F_6=P[x](\{\mathbf{u}_1,\mathbf{u}_3\})$, $F_7=P[x](\{\mathbf{u}_1,\mathbf{u}_2\})$ and $F_8=S$.

Lemma 4.29. *Let S be an affine $P[x]$-semimodule. A subset F of S is a face if and only if $k[S\backslash F]$ is a $P[\sigma]$-prime ideal of $k[S]$.*

Proof. Let $I=k[S\backslash F]:=\bigoplus_{\mathbf{u}\in S\backslash F}k\chi^{\mathbf{u}}$.

For the necessity, suppose that F is a face of S. Note that F can be viewed as a face of S as an $\mathbb{N}[x]$-semimodule, so by [5, Lemma 6.8], I is a σ-prime ideal. We need to show that I is a $P[\sigma]$-ideal. Suppose $\chi^{\mathbf{u}}f\in I$ and $f=\sum_i\alpha_i\chi^{\mathbf{u}_i}$, then $\chi^{\mathbf{u}}f=\sum_i\alpha_i\chi^{\mathbf{u}+\mathbf{u}_i}$. Thus $\mathbf{u}+\mathbf{u}_i\in S\backslash F$. For $g\in P[x]^*$, $\chi^{g\mathbf{u}}f\in I$ is equivalent to $g\mathbf{u}+\mathbf{u}_i\in S\backslash F$ for all i. Suppose to the contrary that $g\mathbf{u}+\mathbf{u}_i\in F$ for some i. Because F is a face of S, it follows that $\mathbf{u},\mathbf{u}_i\in F$ and hence $\mathbf{u}+\mathbf{u}_i\in F$, which is a contradictory.

For the sufficiency, suppose that I is a $P[\sigma]$-prime ideal of $k[S]$. Since I is a $P[\sigma]$-ideal, we have $\mathbf{u}_1\in S\backslash F$ or $\mathbf{u}_2\in S\backslash F$ implies $\mathbf{u}_1+\mathbf{u}_2\in S\backslash F$, and for $g\in P[x]^*$, $\mathbf{u}\in S\backslash F$ implies $g\mathbf{u}\in S\backslash F$. As a consequence, $\mathbf{u}_1+\mathbf{u}_2\in F$ implies $\mathbf{u}_1,\mathbf{u}_2\in F$, and for $g\in P[x]^*$, $g\mathbf{u}\in F$ implies $\mathbf{u}\in F$. Moreover, since I is prime, we have $\mathbf{u}_1+\mathbf{u}_2\in S\backslash F$ implies $\mathbf{u}_1\in S\backslash F$ or $\mathbf{u}_2\in S\backslash F$. As a consequence, $\mathbf{u}_1,\mathbf{u}_2\in F$ implies $\mathbf{u}_1+\mathbf{u}_2\in F$. Since I is perfect, we have for $g\in P[x]^*$, $g\mathbf{u}\in S\backslash F$ implies $\mathbf{u}\in S\backslash F$. As a consequence, for $g\in P[x]^*$, $\mathbf{u}\in F$ implies $g\mathbf{u}\in F$. Put all above together and it follows that F is a face of S. □

Let S be an affine $P[x]$-semimodule. Let $X=\mathrm{Spec}^{P[\sigma]}(k[S])$ be an affine toric $P[\sigma]$-variety and T the σ-torus of X. A $P[\sigma]$-subvariety Y of X is said to be *invariant* under the action of T if $T\cdot Y\subseteq Y$. For a face F of S, let $Y=\mathrm{Spec}^{P[\sigma]}(k[F])$. Without loss of generality, assume that $S=P[x](\{\mathbf{u}_1,\ldots,\mathbf{u}_m\})$ and $F=P[x](\{\mathbf{u}_1,\ldots,\mathbf{u}_r\})$. We always view Y as a $P[\sigma]$-subvariety of X through the embedding $j\colon Y\to X,\gamma\in Y(K)\mapsto(\gamma(\mathbf{u}_1),\ldots,\gamma(\mathbf{u}_r),0,\ldots,0)\in X(K)$ for each $K\in\mathscr{E}_k$. The following theorem gives a description for irreducible invariant $P[\sigma]$-subvarieties of X in terms of the faces of S.

Theorem 4.30. *Let S be an affine $P[x]$-semimodule. Let $X=\mathrm{Spec}^{P[\sigma]}(k[S])$ be an affine toric $P[\sigma]$-variety and T the σ-torus of X. Then the irreducible invariant $P[\sigma]$-subvarieties of X under the action of T are in an inclusion-preserving bijection with the faces of S. More precisely, if we denote the irreducible invariant $P[\sigma]$-subvariety corresponding to the face F by $D(F)$, then $D(F)$ is defined by the $P[\sigma]$-ideal $k[S\backslash F]=\bigoplus_{u\in S\backslash F}k\chi^{\mathbf{u}}$ and the $P[\sigma]$-coordinate ring of $D(F)$ is $k[F]=\bigoplus_{u\in F}k\chi^{\mathbf{u}}$.*

Proof. For a face F of S, let $Y=\mathrm{Spec}^{P[\sigma]}(k[F])$. It is clear that Y is invariant under the action of T. The defining ideal of Y is $I=k[S\backslash F]$. Hence by Lemma 4.29, Y is irreducible.

On the other hand, suppose that Y is an irreducible invariant $P[\sigma]$-subvariety of X and is defined by the $P[\sigma]$-ideal I. Then $k\{Y\}=k[S]/I$. By definition, Y is invariant under the σ-torus action if and only if the action of T on X induces an action on Y, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} k[S] & \xrightarrow{\phi} & k[M]\otimes k[S] \\ \downarrow & & \downarrow \\ k\{Y\} & \longrightarrow & k[M]\otimes k\{Y\} \end{array}$$

where $M=S^{md}$. Since $k[M]\otimes k\{Y\}=k[M]\otimes(k[S]/I)\simeq k[M]\otimes k[S]/k[M]\otimes I$, we must have $\phi(I)\subseteq k[M]\otimes I$. As in the proof of Theorem 4.19, this is equivalent to the fact that I is an M-graded ideal of $k[S]$, i.e. we can write $I=\oplus_{u\in S'}k\chi^{\mathbf{u}}$, where S' is a subset of S. Since I is a $P[\sigma]$-prime ideal, then by Lemma 4.29, $F=S\backslash S'$ is a face of S. Moreover, since $I=k[S\backslash F]$, we have $k\{Y\}=k[S]/I=k[F]$. □

Remark 4.31. Suppose that F is a face of S. Note that for $K \in \mathscr{E}_k$, an element $\gamma\colon S \to K$ of $X(K)$ lies in $D(F)(K)$ if and only if $\gamma(S\backslash F) = 0$.

Suppose that X is an affine toric $P[\sigma]$-variety with σ-torus T. By Theorem 4.19, for each $K \in \mathscr{E}_k$, $T(K)$ has a group action on $X(K)$, so we have orbits of $T(K)$ in $X(K)$ under the action. To construct a correspondence between orbits and faces, we need a new kind of affine $P[x]$-semimodules. Suppose that S is an affine $P[x]$-semimodule, we say that S is *face-saturated* if for any face F of S, a morphism $\varphi\colon F \to K^*$ can be extended to a morphism $\widetilde{\varphi}\colon S \to K^*$ for any $K \in \mathscr{E}_k$. A necessary condition for S to be face-saturated is that for any face F of S, F^{md} is $P[x]$-saturated in S^{md}, that is, for any $g \in P[x]^*$ and $\mathbf{u} \in S^{md}$, $g\mathbf{u} \in F^{md}$ implies $\mathbf{u} \in F^{md}$.

Example 4.32. Let $S = P[x](\{(2,0),(1,1),(0,1)\})$ and $F = P[x]((2,0))$ a face of S. Then $(1,0) \in S^{md}$. Since $(1,0) \notin F$ and $2(1,0) \in F$, S is not face-saturated.

We have the following Orbit-Face correspondence theorem.

Theorem 4.33. *Suppose that S is a face-saturated affine $P[x]$-semimodule. Let $X = \mathrm{Spec}^{P[\sigma]}(k[S])$ be the affine toric $P[\sigma]$-variety associated with S and T the σ-torus of X. Then for each $K \in \mathscr{E}_k$, there is a one-to-one correspondence between the orbits of $T(K)$ in $X(K)$ and the faces of S.*

Proof. The proof is similar to [5, Theorem 6.11]. □

5 Projective Toric $P[\sigma]$-Varieties

In this section, we will define projective toric $P[\sigma]$-varieties.

5.1 Projective $P[\sigma]$-Varieties

Let k be a σ-field. A *σ-projective (m-)space* over k is a functor $\mathbb{P}^m = (\mathbb{A}^{m+1}\backslash\{\mathbf{0}\})/\mathbb{A}^*$ from the category of σ-field extensions of k to the category of sets given by $\mathbb{P}^m(K) = (K^{m+1}\backslash\{\mathbf{0}\})/K^*$ for $K \in \mathscr{E}_k$, where K^* acts via homotheties, i.e. $\lambda \cdot (a_0,\ldots,a_m) = (\lambda a_0,\ldots,\lambda a_m)$ for $\lambda \in K^*$ and $(a_0,\ldots,a_m) \in K^{m+1}$.

Definition 5.1. A $P[\sigma]$-polynomial $f \in k\{y_0,\ldots,y_m\}^{P[\sigma]}$ is said to be *transformally homogeneous* if for a new σ-indeterminate λ, there exists a $P[\sigma]$-monomial $M(\lambda)$ in λ such that $f(\lambda y_0,\ldots,\lambda y_m) = M(\lambda)f(y_0,\ldots,y_m)$. A $P[\sigma]$-ideal is *homogeneous* if it can be generated by a set of transformally homogeneous $P[\sigma]$-polynomials.

Definition 5.2. Suppose F is a set of transformally homogeneous $P[\sigma]$-polynomials in $k\{y_0,\ldots,y_m\}^{P[\sigma]}$. The *projective $P[\sigma]$-variety* over k defined by F is a subfunctor of $\mathbb{P}^m$ given by $\mathbb{V}_K(F) = \{a \in \mathbb{P}^m(K) \mid f(a) = 0, \forall f \in F\}$ for each $K \in \mathscr{E}_k$.

If X is a projective $P[\sigma]$-variety, then the ideal $\mathbb{I}(X)$ generated by all transformally homogeneous $P[\sigma]$-polynomials vanishing on X is called the *defining ideal* or the *vanishing ideal* of X and

$$k\{X\} := k\{y_0,\ldots,y_m\}^{P[\sigma]}/\mathbb{I}(X)$$

is called the *homogeneous $P[\sigma]$-coordinate ring* of X.

Suppose that X is a projective $P[\sigma]$-variety. A $P[\sigma]$-ideal in $k\{X\}$ is *homogeneous* if it can be generated by a set of images of transformally homogeneous $P[\sigma]$-polynomials. Let $\mathrm{Proj}^{P[\sigma]}(k\{X\})$ be the set of all homogeneous $P[\sigma]$-prime ideals of $k\{X\}$ except $\{\bar{y_0},\ldots,\bar{y_m}\}^{P[\sigma]}$ ($\bar{y_i}$ denote the coordinate functions on X, $0 \leqslant i \leqslant m$). Let $F \subseteq k\{X\}$. We set

$$\mathscr{V}(F) := \{\mathfrak{p} \in \mathrm{Proj}^{P[\sigma]}(k\{X\}) \mid F \subseteq \mathfrak{p}\} \subseteq \mathrm{Proj}^{P[\sigma]}(k\{X\}).$$

The following lemma is easy to check.

Lemma 5.3. *Let X be a projective k-$P[\sigma]$-variety and $F, G, F_i \subseteq k\{X\}$. Then*

1. *$\mathscr{V}(0) = \mathrm{Proj}^{P[\sigma]}(k\{X\})$ and $\mathscr{V}(k\{X\}) = \varnothing$;*

2. $\mathscr{V}(F) \cup \mathscr{V}(G) = \mathscr{V}(FG)$;

3. $\bigcap_i \mathscr{V}(F_i) = \mathscr{V}(\bigcup_i F_i)$.

Lemma 5.3 shows that $\mathrm{Proj}^{P[\sigma]}(k\{X\})$ is a topological space with closed sets of the forms $\mathscr{V}(F), F \subseteq k\{X\}$.

Definition 5.4. Let X be a projective k-$P[\sigma]$-variety. Then the *topological space* of X is $\mathrm{Proj}^{P[\sigma]}(k\{X\})$ equipped with the above topology.

5.2 $\mathbb{Z}[x]$-Lattice Points and Projective Toric $P[\sigma]$-Varieties

Note that $\mathbb{P}^{m-1}$ is a toric $P[\sigma]$-variety with the σ-torus

$$\begin{aligned} T_{\mathbb{P}^{m-1}} &= \mathbb{P}^{m-1} \backslash \mathbb{V}(y_0 \dots y_{m-1}) = \{(a_0, \dots, a_{m-1}) \in \mathbb{P}^{m-1} \mid a_0 \dots a_{m-1} \neq 0\} \\ &= \{(1, t_1, \dots, t_{m-1}) \in \mathbb{P}^{m-1} \mid t_1, \dots, t_{m-1} \in \mathbb{A}^*\} \simeq (\mathbb{A}^*)^{m-1}. \end{aligned}$$

The action of $T_{\mathbb{P}^{m-1}}$ on itself clearly extends to an action on $\mathbb{P}^{m-1}$, making $\mathbb{P}^{m-1}$ a toric $P[\sigma]$-variety. To describe the character lattice of $T_{\mathbb{P}^{m-1}}$, let us consider the exact sequence of σ-tori

$$1 \longrightarrow \mathbb{A}^* \longrightarrow (\mathbb{A}^*)^m \xrightarrow{\pi} T_{\mathbb{P}^{m-1}} \longrightarrow 1. \tag{5.1}$$

Applying the functor $\mathrm{Hom}(-, \mathbb{A}^*)$ to (5.1) yields

$$0 \longrightarrow \mathrm{Hom}(T_{\mathbb{P}^{m-1}}, \mathbb{A}^*) \longrightarrow \mathrm{Hom}((\mathbb{A}^*)^m, \mathbb{A}^*) \simeq \mathbb{Z}[x]^m \xrightarrow{\iota} \mathrm{Hom}(\mathbb{A}^*, \mathbb{A}^*) \simeq \mathbb{Z}[x] \longrightarrow 0, \tag{5.2}$$

where $\iota((a_0, \dots, a_{m-1})) = \sum_{i=0}^{m-1} a_i$ for $(a_0, \dots, a_{m-1}) \in \mathbb{Z}[x]^m$. Thus the character lattice of $T_{\mathbb{P}^{m-1}}$ is

$$\mathscr{M}_{m-1} = \mathrm{Hom}(T_{\mathbb{P}^{m-1}}, \mathbb{A}^*) \simeq \{(a_0, \dots, a_{m-1}) \in \mathbb{Z}[x]^m \mid \sum_{i=0}^{m-1} a_i = 0\}.$$

Let $U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$ and $\mathbb{T} = (t_1, \dots, t_n)$ a set of σ-indeterminates. In Section 5, we have defined the affine toric $P[\sigma]$-variety associated with U as the closure of the image of the following map

$$\theta^U \colon (\mathbb{A}^*)^n \longrightarrow \mathbb{A}^m, \mathbb{T} \mapsto \mathbb{T}^U = (\mathbb{T}^{\mathbf{u}_1}, \dots, \mathbb{T}^{\mathbf{u}_m}).$$

To get a projective toric $P[\sigma]$-variety, we regard θ^U as a map to $(\mathbb{A}^*)^m$ and compose with the homomorphism $\pi \colon (\mathbb{A}^*)^m \to T_{\mathbb{P}^{m-1}}$ to obtain

$$(\mathbb{A}^*)^n \xrightarrow{\theta^U} (\mathbb{A}^*)^m \xrightarrow{\pi} T_{\mathbb{P}^{m-1}} \subseteq \mathbb{P}^{m-1}. \tag{5.3}$$

Definition 5.5. Given a finite set $U \subset \mathbb{Z}[x]^n$, the *projective toric $P[\sigma]$-variety* $\mathscr{Y}_U$ is the closure in $\mathbb{P}^{m-1}$ of the image of the map $\pi \circ \theta^U$ from (5.3) under the topology in Definition 5.4.

For a finite set $U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, we have defined the affine toric $P[\sigma]$-variety $\mathscr{X}_U$ in Section 5 and the projective toric $P[\sigma]$-variety $\mathscr{Y}_U$ in Definition 5.5. The following proposition reveals the relationship between $\mathscr{X}_U$ and $\mathscr{Y}_U$. Let $M = \mathbb{Z}[x](U)$ and $L = \mathrm{Syz}(U)$, and we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}[x]^m \xrightarrow{\alpha} M \longrightarrow 0, \tag{5.4}$$

where α maps the standard base vector $\mathbf{e}_i$ to $\mathbf{u}_i$, $i = 1, \dots, m$. The vanishing ideal of $\mathscr{X}_U$ is the binomial $P[\sigma]$-ideal

$$J_L = [\mathbb{Y}^{\mathbf{u}} - \mathbb{Y}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in P[x]^m \text{ with } \mathbf{u} - \mathbf{v} \in L].$$

Proposition 5.6. For a finite set $U \subset \mathbb{Z}[x]^n$, the followings are equivalent:

(a) $\mathbb{I}(\mathscr{X}_U) = J_L = \mathbb{I}(\mathscr{Y}_U)$;

(b) J_L is homogeneous;

(c) There exists a vector $\mathbf{v} \in \mathbb{Z}[x]^n$ and $g \in \mathbb{Z}[x]$ such that $\langle \mathbf{u}_i, \mathbf{v} \rangle = g$ for all $\mathbf{u}_i \in U$.

Proof. (a)$\Leftrightarrow$(b) is easy from the definitions.

(b)$\Rightarrow$(c). Assume that J_L is homogeneous and take $\mathbb{Y}^{\mathbf{u}} - \mathbb{Y}^{\mathbf{v}} \in J_L$ for $\mathbf{u} = (u_i)_{i=1}^m, \mathbf{v} = (v_i)_{i=1}^m \in P[x]^m$ and $\mathbf{u} - \mathbf{v} \in L$. If $\sum_{i=1}^m u_i \neq \sum_{i=1}^m v_i$, then $\mathbb{Y}^{\mathbf{u}}, \mathbb{Y}^{\mathbf{v}} \in J_L$ which is impossible. So $\sum_{i=1}^m u_i = \sum_{i=1}^m v_i$. It follows $\mathbf{u} \cdot (1, \ldots, 1) = 0$ for all $\mathbf{u} \in L$. Now apply the functor $\mathrm{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z}[x])$ to (5.4) and we obtain an exact sequence

$$M^* \longrightarrow \mathbb{Z}[x]^m \longrightarrow \mathrm{Hom}_{\mathbb{Z}[x]}(L, \mathbb{Z}[x]) \longrightarrow 0, \tag{5.5}$$

where $M^* = \mathrm{Hom}_{\mathbb{Z}[x]}(M, \mathbb{Z}[x])$. The above argument shows that $(1, \ldots, 1) \in \mathbb{Z}[x]^m$ maps to zero in $\mathrm{Hom}_{\mathbb{Z}[x]}(L, \mathbb{Z}[x])$ and hence there exists $\varphi \in M^*$ such that $\varphi(\mathbf{u}_i) = 1$ for all i. By Lemma 2.13, there exists a vector $\mathbf{v} \in \mathbb{Z}[x]^n$ and $g \in \mathbb{Z}[x]$ such that $g\varphi = \varphi_{\mathbf{v}} = \langle -, \mathbf{v} \rangle$. In particular, $\langle \mathbf{u}_i, \mathbf{v} \rangle = g$ for all $\mathbf{u}_i \in U$.

(c)$\Rightarrow$(b). The vector $\mathbf{v} \in \mathbb{Z}[x]^n$ gives $\varphi_{\mathbf{v}} = \langle -, \mathbf{v} \rangle \in M^*$ such that $\varphi_{\mathbf{v}}(\mathbf{u}_i) = g$ for all i. From (5.5), $(g, \ldots, g)$ maps to zero in $\mathrm{Hom}_{\mathbb{Z}[x]}(L, \mathbb{Z}[x])$. It follows that for any $\mathbf{u} \in L$, $\sum_{i=1}^m g u_i = 0$ and hence $\sum_{i=1}^m u_i = 0$. So J_L is homogeneous. □

Given $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, we set

$$\mathbb{Z}[x]'(U) := \{\sum_{i=1}^m a_i \mathbf{u}_i \mid a_i \in \mathbb{Z}[x], \sum_{i=1}^m a_i = 0\}.$$

Proposition 5.7. Let $\mathscr{Y}_U$ be the projective toric $P[\sigma]$-variety defined by U. Then

(a) The character lattice of the σ-torus of $\mathscr{Y}_U$ is $\mathbb{Z}[x]'(U)$.

(b) The σ-dimension of $\mathscr{Y}_U$ is the dimension of the smallest affine subspace of $\mathbb{Q}(x)^m$ containing U. More concretely,

$$\sigma\text{-}\dim(\mathscr{Y}_U) = \begin{cases} \mathrm{rank}(U) - 1, & \text{if } U \text{ satisfies the conditions of Proposition 5.6;} \\ \mathrm{rank}(U), & \text{otherwise.} \end{cases}$$

Proof. (a) The proof is similar to Proposition 2.1.6(a) in [3, p.58].
(b) By (a), the σ-dimension of $\mathscr{Y}_U$ equals to the rank of $\mathbb{Z}[x]'(U)$. Let $U' = \{\mathbf{u}_2 - \mathbf{u}_1, \ldots, \mathbf{u}_m - \mathbf{u}_1\}$. It is easy to check that $\mathbb{Z}[x]'(U) = \mathbb{Z}[x](U')$. So $\mathrm{rank}(\mathbb{Z}[x]'(U)) = \mathrm{rank}(U')$ and the conclusions of (b) then follow. □

In the following we show that a projective toric $P[\sigma]$-variety is actually covered by a series of affine toric $P[\sigma]$-varieties. Given a finite set $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$, let $\mathscr{Y}_U \subseteq \mathbb{P}^{m-1}$ be the projective toric $P[\sigma]$-variety defined by U and $T_{\mathscr{Y}_U}$ the σ-torus. For $1 \leqslant i \leqslant m$, let $O_i = \mathbb{P}^{m-1} \backslash \mathbb{V}(y_i)$ which is an affine open subset containing the σ-torus $T_{\mathbb{P}^{m-1}}$. We have

$$T_{\mathscr{Y}_U} = \mathscr{Y}_U \cap T_{\mathbb{P}^{m-1}} \subseteq \mathscr{Y}_U \cap O_i.$$

Since $\mathscr{Y}_U$ is the closure of $T_{\mathscr{Y}_U}$ in $\mathbb{P}^{m-1}$, it follows that $\mathscr{Y}_U \cap O_i$ is the closure of $T_{\mathscr{Y}_U}$ in $O_i \simeq \mathbb{A}^{m-1}$. Thus $\mathscr{Y}_U \cap O_i$ is an affine toric $P[\sigma]$-variety. We will determinate the affine $P[x]$-semimodule associated with $\mathscr{Y}_U \cap O_i$. $O_i \simeq \mathbb{A}^{m-1}$ is given by

$$(a_1, \ldots, a_m) \mapsto (a_1/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_m/a_i).$$

Combining this with the map (5.3), we see that $\mathscr{Y}_U \cap O_i$ is the closure of the image of the map $(\mathbb{A}^*)^m \to \mathbb{A}^{m-1}$ given by

$$\mathbb{T} \mapsto (\mathbb{T}^{\mathbf{u}_1 - \mathbf{u}_i}, \ldots, \mathbb{T}^{\mathbf{u}_{i-1} - \mathbf{u}_i}, \mathbb{T}^{\mathbf{u}_{i+1} - \mathbf{u}_i}, \ldots, \mathbb{T}^{\mathbf{u}_m - \mathbf{u}_i}). \tag{5.6}$$

If we set $U_i = U - \mathbf{u}_i := \{\mathbf{u}_j - \mathbf{u}_i \mid j \neq i, 1 \leqslant j \leqslant m\}$ and $S_i = P[x](U_i)$, it follows

$$\mathscr{Y}_U \cap O_i = X_{U_i} = \mathrm{Spec}^{P[\sigma]}(k[S_i]).$$

So we have the following proposition.

Proposition 5.8. Let $\mathscr{Y}_U \subseteq \mathbb{P}^{m-1}$ for $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{Z}[x]^n$. Then the affine piece $\mathscr{Y}_U \cap O_i$ is the affine toric $P[\sigma]$-variety

$$\mathscr{Y}_U \cap O_i = \mathscr{X}_{U_i} = \mathrm{Spec}^{P[\sigma]}(k[S_i]),$$

where $U_i = U - \mathbf{u}_i = \{\mathbf{u}_j - \mathbf{u}_i \mid j \neq i, 1 \leqslant j \leqslant m\}$ and $S_i = P[x](U_i)$, $i = 1, \ldots, m$.

Besides describing the affine pieces $\mathscr{Y}_U \cap O_i$ of $\mathscr{Y}_U \subseteq \mathbb{P}^{m-1}$, we can also describe how they patch together. When $i \neq j$, $\mathscr{Y}_U \cap O_i \cap O_j$ consists of all points of $\mathscr{Y}_U \cap O_i$ where $y_j/y_i \neq 0$. By (5.6), this means those points where $\chi^{\mathbf{u}_j - \mathbf{u}_i} \neq 0$. Thus

$$\begin{aligned}\mathscr{Y}_U \cap O_i \cap O_j &= \mathrm{Spec}^{P[\sigma]}(k[S_i])_{\chi^{\mathbf{u}_j - \mathbf{u}_i}} = \mathrm{Spec}^{P[\sigma]}(k[S_i]_{\chi^{\mathbf{u}_j - \mathbf{u}_i}}) \\ &= \mathrm{Spec}^{P[\sigma]}(k[S_i + \mathbb{Z}[x](\mathbf{u}_i - \mathbf{u}_j)]) \subseteq \mathscr{Y}_U \cap O_i.\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{Y}_U \cap O_i \cap O_j &= \mathrm{Spec}^{P[\sigma]}(k[S_j])_{\chi^{\mathbf{u}_i - \mathbf{u}_j}} = \mathrm{Spec}^{P[\sigma]}(k[S_j]_{\chi^{\mathbf{u}_i - \mathbf{u}_j}}) \\ &= \mathrm{Spec}^{P[\sigma]}(k[S_j + \mathbb{Z}[x](\mathbf{u}_j - \mathbf{u}_i)]) \subseteq \mathscr{Y}_U \cap O_j.\end{aligned}$$

Remark 5.9. One can check that $S_i + \mathbb{Z}[x](\mathbf{u}_i - \mathbf{u}_j) = S_j + \mathbb{Z}[x](\mathbf{u}_j - \mathbf{u}_i)$.

6 Abstract Toric $P[\sigma]$-Varieties

In this section, we will define abstract toric $P[\sigma]$-varieties through gluing affine toric $P[\sigma]$-varieties along open subsets and generalize the irreducible invariant $P[\sigma]$-subvarieties-faces correspondence to abstract toric $P[\sigma]$-varieties.

6.1 Gluing Together Affine $P[\sigma]$-Varieties

Suppose that we have a finite collection $\{V_\alpha\}_\alpha$ of affine $P[\sigma]$-varieties and for all pairs α, β we have open subsets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\beta\alpha} \colon V_{\beta\alpha} \simeq V_{\alpha\beta}$ satisfying the following compatibility conditions:

- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all α, β;
- $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$ for all α, β, γ.

Now we can glue together $\{V_\alpha\}_\alpha$ along open subsets $V_{\alpha\beta}$ through isomorphisms $g_{\beta\alpha}$, and denote it by X.

Definition 6.1. The above X is called an *abstract $P[\sigma]$-variety*. Its open sets are those subsets that restrict to open subsets in each V_α. Its closed sets are called *$P[\sigma]$-subvarieties* of X. We say that X is *irreducible* if it is not the union of two proper $P[\sigma]$-subvarieties.

6.2 The Toric $P[\sigma]$-Variety of a Fan

Now we give the definition of abstract toric $P[\sigma]$-varieties.

Definition 6.2. An *(abstract) toric $P[\sigma]$-variety* is an irreducible abstract $P[\sigma]$-variety X containing a σ-torus T as an open subset such that the action of T on itself extends to a σ-algebraic group action of T on X.

It is clear that both affine toric $P[\sigma]$-varieties and projective toric $P[\sigma]$-varieties we have defined in the previous sections are abstract toric $P[\sigma]$-varieties.

We will construct abstract toric $P[\sigma]$-varieties from fans. First we give the definition of a fan.

Definition 6.3. Let $\{S_i\}_i$ be a finite collection of affine $P[x]$-semimodules in $\mathbb{Z}[x]^n$. We say that $\{S_i\}$ is *compatible* if it satisfies:

(a) there exists a $\mathbb{Z}[x]$-lattice M such that $S_i^{md} = M$ for all i;

(b) for all pairs (i, j), there exists $\mathbf{u} \in S_i$ such that $-\mathbf{u} \in S_j$ and $S_i + \mathbb{Z}[x](-\mathbf{u}) = S_j + \mathbb{Z}[x](\mathbf{u})$;

(c) for all triples (i,j,k), by (b), there exist $\mathbf{u}\in S_i, \mathbf{v}\in S_j, \mathbf{w}\in S_k$ such that $S_i+\mathbb{Z}[x](-\mathbf{u}) = S_j+\mathbb{Z}[x](\mathbf{u}), S_j+\mathbb{Z}[x](-\mathbf{v}) = S_k+\mathbb{Z}[x](\mathbf{v}), S_k+\mathbb{Z}[x](-\mathbf{w}) = S_i+\mathbb{Z}[x](\mathbf{w})$. Such $\mathbf{u},\mathbf{v},\mathbf{w}$ satisfy $S_i+\mathbb{Z}[x](-\mathbf{u})+\mathbb{Z}[x](\mathbf{w}) = S_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{v}) = S_k+\mathbb{Z}[x](\mathbf{v})+\mathbb{Z}[x](-\mathbf{w})$.

Definition 6.4. A *fan* Σ is a finite collection of affine $P[x]$-semimodules $\{S_i\}_i$ which is compatible. If Σ is a fan, we write $\Sigma^{md} = S_i^{md}$ for any i and define $\mathrm{rank}(\Sigma) = \mathrm{rank}(\Sigma^{md})$.

Example 6.5. Let $U = \{\mathbf{u}_1,\dots,\mathbf{u}_m\}\subset\mathbb{Z}[x]^n$ and $S_i = \mathbb{Z}[x](U-\mathbf{u}_i), i=1,\dots,m$. One can check that $\Sigma = \{S_i\}_{i=1}^m$ satisfies the above compatible conditions and thus is a fan.

We now show that how we can construct an abstract toric $P[\sigma]$-variety from a fan. Let $\Sigma = \{S_i\}_i$ be a fan. By Theorem 4.11, each S_i in Σ gives an affine toric $P[\sigma]$-variety $X_i = \mathrm{Spec}^{P[\sigma]}(k[S_i])$. Let S_i and S_j be two different affine $P[x]$-semimodules in Σ, then by Definition 6.3(b), there exists $\mathbf{u}\in\Sigma^{md}$, such that $k[S_i]_{\chi^{\mathbf{u}}} = k[S_j]_{\chi^{-\mathbf{u}}}$, so we have an isomorphism

$$g_{ji}\colon (X_i)_{\chi^{\mathbf{u}}} \simeq (X_j)_{\chi^{-\mathbf{u}}}, \tag{6.1}$$

which is the identity map. For any distinct i,j,k, there exist $\mathbf{u},\mathbf{v},\mathbf{w}\in\Sigma^{md}$, such that

$$\begin{aligned}(X_i)_{\chi^{\mathbf{u}}}\cap(X_i)_{\chi^{-\mathbf{w}}} &= \mathrm{Spec}^{P[\sigma]}(S_i+\mathbb{Z}[x](-\mathbf{u})+\mathbb{Z}[x](\mathbf{w})),\\ (X_j)_{\chi^{-\mathbf{u}}}\cap(X_j)_{\chi^{\mathbf{v}}} &= \mathrm{Spec}^{P[\sigma]}(S_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{v})),\\ (X_k)_{\chi^{-\mathbf{v}}}\cap(X_k)_{\chi^{\mathbf{w}}} &= \mathrm{Spec}^{P[\sigma]}(S_k+\mathbb{Z}[x](\mathbf{v})+\mathbb{Z}[x](-\mathbf{w})),\end{aligned}$$

and by Definition 6.3(c),

$$(X_i)_{\chi^{\mathbf{u}}}\cap(X_i)_{\chi^{-\mathbf{w}}} = (X_j)_{\chi^{-\mathbf{u}}}\cap(X_j)_{\chi^{\mathbf{v}}} = (X_k)_{\chi^{-\mathbf{v}}}\cap(X_k)_{\chi^{\mathbf{w}}}.$$

So the compatibility conditions for gluing the affine toric $P[\sigma]$-varieties X_i along the open subsets $(X_i)_{\chi^{\mathbf{u}}}$ are satisfied. Hence we obtain an abstract $P[\sigma]$-variety X_Σ associated with the fan Σ.

Theorem 6.6. *Let $\Sigma = \{S_i\}_i$ be a fan in $\mathbb{Z}[x]^n$. Then the abstract $P[\sigma]$-variety X_Σ constructed above is a toric $P[\sigma]$-variety.*

Proof. Let $M = \Sigma^{md}$ and $T_\Sigma = \mathrm{Spec}^{P[\sigma]}(k[M])$. Then $T_\Sigma\subseteq X_i = \mathrm{Spec}^{P[\sigma]}(k[S_i])$ as a σ-torus for all i. These σ-tori are all identified by the gluing, so $T_\Sigma\subseteq X_\Sigma$ as an open subset of X_Σ. For each i, T_Σ has an action on X_i. The gluing isomorphisms g_{ji} in (6.1) are identity maps on each $X_i\cap X_j$, so the actions are compatible on each $X_i\cap X_j$, and patch together to give a σ-algebraic group action of T_Σ on X_Σ. X_Σ is irreducible since all X_i are irreducible affine toric $P[\sigma]$-varieties. So by Definition 6.2, X_Σ is a toric $P[\sigma]$-variety. □

Example 6.7. Let $U = \{\mathbf{u}_1,\dots,\mathbf{u}_m\}\subset\mathbb{Z}[x]^n$ and $S_i = \mathbb{Z}[x](U-\mathbf{u}_i), i=1,\dots,m$. In Example 6.5, we see that $\Sigma = \{S_i\}_{i=1}^m$ is a fan. The projective toric $P[\sigma]$-variety $\mathscr{Y}_U$ defined by U is an abstract toric $P[\sigma]$-variety associated with the fan Σ.

The irreducible invariant $P[\sigma]$-subvarieties-faces correspondence (Theorem 4.30) still applies to abstract toric $P[\sigma]$-varieties constructed from fans through considering the gluing.

Suppose $\Sigma = \{S_i\}_i$ is a fan in $\mathbb{Z}[x]^n$. Let

$$F_\Sigma := \{F \mid F\preceq S_i, S_i\in\Sigma\}$$

be the set of faces of affine $P[x]$-semimodules in Σ, and

$$F_\Sigma(r) := \{F\in F_\Sigma \mid \mathrm{rank}(F) = \mathrm{rank}(\Sigma)-r\}.$$

Define an equivalence relationship in F_Σ as follows: for $F_i\preceq S_i, F_j\preceq S_j$, assume $\mathbf{u}\in S_i$ such that $S_i+\mathbb{Z}[x](-\mathbf{u}) = S_j+\mathbb{Z}[x](\mathbf{u})$. Then $F_i\sim F_j$ if and only if there exists a face F of $S_i+\mathbb{Z}[x](-\mathbf{u})$ such that $F_i^{md} = F_j^{md} = F^{md}$.

Let $L_\Sigma := F_\Sigma/\sim$ and $L_\Sigma(r) := F_\Sigma(r)/\sim$. For $L', L\in L_\Sigma$, $L'\preceq L$ means that there exists a representative F' of L' and a representative F of L such that $F'\preceq F$.

Let us prove some lemmas.

Lemma 6.8. *Let S be an affine $P[x]$-semimodule and F a face of S. Then for any $\mathbf{u} \in F^{md}$, $F+\mathbb{Z}[x](\mathbf{u})$ is a face of $S+\mathbb{Z}[x](\mathbf{u})$.*

Proof. Suppose $\mathbf{a}, \mathbf{b} \in S+\mathbb{Z}[x](\mathbf{u})$ such that $\mathbf{a}+\mathbf{b} \in F+\mathbb{Z}[x](\mathbf{u})$. Write $\mathbf{a} = \mathbf{a}'+g_1\mathbf{u}$, $\mathbf{b} = \mathbf{b}'+g_2\mathbf{u}$, and $\mathbf{a}+\mathbf{b} = \mathbf{c}+g_3\mathbf{u}$, where $\mathbf{a}', \mathbf{b}' \in S, \mathbf{c} \in F, g_1, g_2, g_3 \in \mathbb{Z}[x]$. Then $\mathbf{a}+\mathbf{b} = \mathbf{a}'+\mathbf{b}'+(g_1+g_2)\mathbf{u} = \mathbf{c}+g_3\mathbf{u}$. So $\mathbf{a}'+\mathbf{b}'+(g_1+g_2-g_3)\mathbf{u} = \mathbf{c}$. Since $(g_1+g_2-g_3)\mathbf{u} \in F^{md}$, we can write $(g_1+g_2-g_3)\mathbf{u} = \mathbf{d}-\mathbf{e}, \mathbf{d}, \mathbf{e} \in F$. So $\mathbf{a}'+\mathbf{b}'+\mathbf{d} = \mathbf{c}+\mathbf{e} \in F$. It follows $\mathbf{a}', \mathbf{b}' \in F$. Thus $\mathbf{a}, \mathbf{b} \in F+\mathbb{Z}[x](\mathbf{u})$. Suppose $\mathbf{a} \in S+\mathbb{Z}[x](\mathbf{u})$, $g \in P[x]^*$ such that $g\mathbf{a} \in F+\mathbb{Z}[x](\mathbf{u})$. Write $\mathbf{a} = \mathbf{a}'+g_1\mathbf{u}$, and $g\mathbf{a} = \mathbf{c}+g_2\mathbf{u}$, where $\mathbf{a}' \in S, \mathbf{c} \in F, g_1, g_2 \in \mathbb{Z}[x]$. Then $g\mathbf{a} = g\mathbf{a}'+gg_1\mathbf{u} = \mathbf{c}+g_2\mathbf{u}$. So $g\mathbf{a}'+(gg_1-g_2)\mathbf{u} = \mathbf{c}$. Since $(gg_1-g_2)\mathbf{u} \in F^{md}$, we can write $(gg_1-g_2)\mathbf{u} = \mathbf{d}-\mathbf{e}, \mathbf{d}, \mathbf{e} \in F$. So $g\mathbf{a}'+\mathbf{d} = \mathbf{c}+\mathbf{e} \in F$. It follows $\mathbf{a}' \in F$. Thus $\mathbf{a} \in F+\mathbb{Z}[x](\mathbf{u})$. Hence $F+\mathbb{Z}[x](\mathbf{u})$ is a face of $S+\mathbb{Z}[x](\mathbf{u})$. □

Remark 6.9. From the above lemma, we see that if $\Sigma = \{S_i\}_i$ is a fan and $\mathbf{u} \in S_i$ such that $S_i+\mathbb{Z}[x](-\mathbf{u}) = S_j+\mathbb{Z}[x](\mathbf{u})$, then for $F_i \preceq S_i, F_j \preceq S_j$, $F_i \sim F_j$ if and only if $\mathbf{u} \in F_i^{md} = F_j^{md}$.

Lemma 6.10. *Let F be a face of an affine $P[x]$-semimodule S. Then $F = S \cap F^{md}$.*

Proof. Clearly, $F \subseteq S \cap F^{md}$. We need to show $F \supseteq S \cap F^{md}$. Suppose $S = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\})$ and $F = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_r\})$. If $\mathbf{w} \in S \cap F^{md}$, we can write $\mathbf{w} = \sum_{i=1}^m g_i\mathbf{u}_i = \sum_{i=1}^r f_i\mathbf{u}_r$, where $g_i \in P[x], 1 \leqslant i \leqslant m$ and $f_i \in \mathbb{Z}[x], 1 \leqslant i \leqslant r$. Assume $f_i = (f_i)_+ - (f_i)_-, (f_i)_+, (f_i)_- \in P[x], 1 \leqslant i \leqslant r$. Then we have $\sum_{i=1}^r (g_i+(f_i)_-)\mathbf{u}_i + \sum_{i=r+1}^m g_i\mathbf{u}_i = \sum_{i=1}^r (f_i)_+\mathbf{u}_r \in F$. Since F is a face of S, it follows $\sum_{i=r+1}^m g_i\mathbf{u}_i \in F$ and if $g_i \neq 0$, then $\mathbf{u}_i \in F$, $r+1 \leqslant i \leqslant m$. Because $\mathbf{u}_i \notin F, r+1 \leqslant i \leqslant m$, then $g_i = 0, r+1 \leqslant i \leqslant m$. Thus $\mathbf{w} = \sum_{i=1}^r g_i\mathbf{u}_i \in F$. □

Lemma 6.11. *Let $\Sigma = \{S_i\}_i$ be a fan. Then every $L \in L_\Sigma$ is also a fan.*

Proof. We need to check that L satisfies the three compatible conditions (a), (b), (c) in Definition 6.3. For $F_i, F_j \in L$, assume $F_i \preceq S_i, F_j \preceq S_j$ and $S_i+\mathbb{Z}[x](-\mathbf{u}) = S_j+\mathbb{Z}[x](\mathbf{u})$. Since $F_i \sim F_j$, there exists a face F of $S_i+\mathbb{Z}[x](-\mathbf{u})$ such that $F_i^{md} = F_j^{md} = F^{md}$. So (a) is satisfied. Since F is a face of $S_i+\mathbb{Z}[x](-\mathbf{u})$, we see $\mathbf{u} \in F$ and $\mathbf{u} \in F_i^{md} = F_j^{md}$. To prove (b), because of the symmetry, we just need to show $F_i+\mathbb{Z}[x](-\mathbf{u}) \subseteq F_j+\mathbb{Z}[x](\mathbf{u})$, or $F_i \subseteq F_j+\mathbb{Z}[x](\mathbf{u})$. Since $F_i \subseteq S_i \subseteq S_j+\mathbb{Z}[x](\mathbf{u})$, for $\mathbf{a} \in F_i$, we can write $\mathbf{a} = \mathbf{b}+g\mathbf{u}$, where $\mathbf{b} \in S_j, g \in \mathbb{Z}[x]$. Then $\mathbf{b} = \mathbf{a}-g\mathbf{u} \in F_j^{md} \cap S_j$. Therefore, by Lemma 6.10, $\mathbf{b} \in F_j$. Hence $\mathbf{a} \in F_j+\mathbb{Z}[x](\mathbf{u})$. So (b) is satisfied. To prove (c), suppose $F_i \sim F_j \sim F_k \in L$ and $F_i \preceq S_i, F_j \preceq S_j, F_k \preceq S_k$. Assume $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \Sigma^{md}$ such that $S_i+\mathbb{Z}[x](-\mathbf{u})+\mathbb{Z}[x](-\mathbf{v}) = S_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{w}) = S_k+\mathbb{Z}[x](\mathbf{v})+\mathbb{Z}[x](\mathbf{w})$. As above, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in F_i^{md} = F_j^{md} = F_k^{md}$. For the symmetry, we just need to prove $F_i+\mathbb{Z}[x](-\mathbf{u})+\mathbb{Z}[x](-\mathbf{v}) \subseteq F_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{w})$, or $\mathbb{Z}[x](-\mathbf{v}) \subseteq F_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{w})$. Suppose $g\mathbf{v} \in \mathbb{Z}[x](-\mathbf{v})$. Since $g\mathbf{v} \in S_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{w})$, we can write $g\mathbf{v} = \mathbf{b}+h_1\mathbf{u}+h_2\mathbf{w}$, where $\mathbf{b} \in S_j, h_1, h_2 \in \mathbb{Z}[x]$. Then $\mathbf{b} = g\mathbf{v}-h_1\mathbf{u}-h_2\mathbf{w} \in F_j^{md} \cap S_j$. Therefore, by Lemma 6.10 again, $\mathbf{b} \in F_j$. Hence $g\mathbf{v} \in F_j+\mathbb{Z}[x](\mathbf{u})+\mathbb{Z}[x](-\mathbf{w})$. So (c) is satisfied. Hence L is a fan. □

Now we give the irreducible invariant $P[\sigma]$-subvarieties-faces correspondence theorem.

Theorem 6.12. *Let X_Σ be the toric $P[\sigma]$-variety associated with a fan $\Sigma = \{S_i\}_i$ and assume that T_Σ is the σ-torus of X_Σ. Then there is a one-to-one correspondence between elements of L_Σ and irreducible T_Σ-invariant $P[\sigma]$-subvarieties of X_Σ. Let $L \in L_\Sigma$. If we denote the irreducible T_Σ-invariant $P[\sigma]$-subvariety associated with L by $D(L)$, then $D(L) \simeq X_L$ which is the toric $P[\sigma]$-variety associated with the fan L.*

Proof. For an element $L = \{F_i\}_i$ of L_Σ, each $F_i \preceq S_i$ corresponds to an irreducible T_Σ-invariant $P[\sigma]$-subvariety of $X_i = \mathrm{Spec}^{P[\sigma]}(k[S_i])$. The gluing of X_i induces a gluing of $X_{F_i} = \mathrm{Spec}^{P[\sigma]}(k[F_i])$. The resulted $P[\sigma]$-variety is exactly X_L since L is a fan by Lemma 6.11.

For the converse, suppose that Y is an irreducible T_Σ-invariant $P[\sigma]$-subvariety of X_Σ. Then $Y \cap X_i$ is an irreducible T_Σ-invariant $P[\sigma]$-subvariety of X_i, thus there exists a face F_i of S_i such that $Y \cap X_i = \mathrm{Spec}^{P[\sigma]}(k[F_i])$. The gluing of $Y \cap X_i$ is induced by the gluing of X_i. Therefore if $S_i+\mathbb{Z}[x](-\mathbf{u}) = S_j+\mathbb{Z}[x](\mathbf{u})$, then $F_i+\mathbb{Z}[x](-\mathbf{u}) = F_j+\mathbb{Z}[x](\mathbf{u})$, and $\mathbf{u} \in F_i^{md} = F_j^{md}$. Let $F = F_i+\mathbb{Z}[x](-\mathbf{u})$ which is a face of $S_i+\mathbb{Z}[x](-\mathbf{u})$ by Lemma 6.8. Obviously, $F^{md} = F_i^{md} = F_j^{md}$. Therefore, $F_i \sim F_j$. So $L = \{F_i\}_i$ is an element of L_Σ. □

7 Divisors on Toric $P[\sigma]$-Varieties

In algebraic geometry, the divisor theory is a very useful tool to study the properties of algebraic varieties. In this section, we will define divisors and divisor class modules for toric $P[\sigma]$-varieties by virtue of the irreducible invariant $P[\sigma]$-subvarieties-faces correspondence. Moreover, we will establish connections between the properties of toric $P[\sigma]$-varieties and divisor class modules.

First let us give a description of faces of affine $P[x]$-semimodules using supporting hyperplanes. Let S be an affine $P[x]$-semimodule and $M = S^{md}$. Let $M^* = \mathrm{Hom}_{\mathbb{Z}[x]}(M, \mathbb{Z}[x])$. Given $\varphi \in M^*, \varphi \neq 0$, let

$$H_\varphi := \{\mathbf{u} \in M \mid \varphi(\mathbf{u}) = 0\} \subseteq M$$

which is called the *hyperplane* defined by φ and

$$H_\varphi^+ := \{\mathbf{u} \in M \mid \varphi(\mathbf{u}) \geqslant 0\} \subseteq M$$

which is called the *closed half-space* defined by φ. If $S \subseteq H_\varphi^+$, then H_φ is called a *supporting hyperplane* of S and H_φ^+ is called a *supporting half-space*.

Proposition 7.1. Let S be an affine $P[x]$-semimodule and $M = S^{md}$. Then F is a proper face of S if and only if there exists a $\varphi \in M^*$ such that H_φ^+ is a supporting half-space of S and $F = H_\varphi \cap S$.

Proof. For the necessity, suppose $S = P[x](U) = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}) \subseteq \mathbb{Z}[x]^n$ and $\mathrm{rank}(S) = t$. Let $L = \mathrm{Syz}(U)$ and $\mathrm{rank}(L) = m - t$ by Lemma 2.8. Then the map $\mathbf{e}_i \mapsto \mathbf{u}_i, 1 \leqslant i \leqslant m$ ($\{\mathbf{e}_i\}_{i=1}^m$ is the standard basis of $\mathbb{Z}[x]^m$) gives the following exact sequence:

$$0 \longrightarrow L \longrightarrow \mathbb{Z}[x]^m \longrightarrow M \longrightarrow 0.$$

Without loss of generality, assume $F = P[x](\{\mathbf{u}_1, \ldots, \mathbf{u}_r\})$ and $\mathrm{rank}(F) = s < t$. Let $V = \{\varphi \in M^* \mid F = H_\varphi \cap S\}$. If $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_{m-t}\}$ is a basis of L and regard A as a matrix with columns $\mathbf{a}_i, 1 \leqslant i \leqslant m-t$, then

$$M^* \simeq \{\varphi = (\varphi_1, \ldots, \varphi_m)^\tau \in \mathbb{Z}[x]^m \mid A\varphi = \mathbf{0}\}$$

and

$$V \simeq \{\varphi = (0, \ldots, 0, \varphi_{r+1}, \ldots, \varphi_m)^\tau \in \mathbb{Z}[x]^m \mid A\varphi = \mathbf{0}\}.$$

So V is a free $\mathbb{Z}[x]$-module and $\mathrm{rank}(V) = (m-r) - ((m-t) - (r-s)) = t - s > 0$ by Lemma 2.8. We can assume that A is a trapezoidal matrix. Because F is a face, we can choose $\varphi \in V$ such that $\varphi(\mathbf{u}_i) \neq 0, i = r+1, \ldots, m$. Then $F = H_\varphi \cap S$. We claim that $\varphi(\mathbf{u}_i)$ have the same sign for $i = r+1, \ldots, m$. Otherwise, suppose that $\varphi(\mathbf{u}_i) > 0$ and $\varphi(\mathbf{u}_j) < 0$, $i, j \geqslant r+1$. Then $\varphi(\varphi(\mathbf{u}_i)\mathbf{u}_j - \varphi(\mathbf{u}_j)\mathbf{u}_i) = 0$, thus $\varphi(\mathbf{u}_i)\mathbf{u}_j - \varphi(\mathbf{u}_j)\mathbf{u}_i \in F$ and it follows $\mathbf{u}_i, \mathbf{u}_j \in F$ which is a contradiction. Therefore, without loss of generality, we can assume that $\varphi(\mathbf{u}_i) > 0, i = r+1, \ldots, m$ and hence H_φ^+ is a supporting half-space of S.

For the sufficiency, suppose that H_φ^+ is a supporting half-space of S and $F = H_\varphi \cap S$, where $\varphi \in M^*, \varphi \neq 0$. It is clear that F is a $P[x]$-semimodule. Let $\mathbf{u}_1, \mathbf{u}_2 \in S, g_1, g_2 \in P[x]^*$ such that $g_1\mathbf{u}_1 + g_2\mathbf{u}_2 \in F$. Then $\varphi(g_1\mathbf{u}_1 + g_2\mathbf{u}_2) = g_1\varphi(\mathbf{u}_1) + g_2\varphi(\mathbf{u}_2) = 0$. Since $\mathbf{u}_1, \mathbf{u}_2 \in S$, $\varphi(\mathbf{u}_1) \geqslant 0$ and $\varphi(\mathbf{u}_2) \geqslant 0$. We must have $\varphi(\mathbf{u}_1) = \varphi(\mathbf{u}_2) = 0$. It follows $\mathbf{u}_1, \mathbf{u}_2 \in F$ and hence F is a face as desired. □

Suppose that S is an affine $P[x]$-semimodule. If F is a proper face of S, let $V(F) := \{\varphi \in M^* \mid F = H_\varphi \cap S\}$. From the proof of the above proposition, we know that $V(F)$ is a free $\mathbb{Z}[x]$-module and $\mathrm{rank}(V(F)) = \mathrm{rank}(S) - \mathrm{rank}(F)$. In particular, if F is a facet of S, then $V(F)$ is a free $\mathbb{Z}[x]$-module of rank one which has a basis $\{\varphi_F\}$ with $S \subseteq H_{\varphi_F}^+$. In this case, we call φ_F the *standard normal vector* of F.

Let $\Sigma = \{S_i\}_i$ be a fan and $M = \Sigma^{md}$. We will define a valuation on $k(M)$ for every facet of S_i. First consider the affine case. Let S be an affine $P[x]$-semimodule and F a facet of S. Assume that φ_F is the standard normal vector of F. Define a *valuation* ν_F on $k[S]$ as follows:

$$\nu_F \colon k[S] \to \mathbb{Z}[x], f = \sum_{\mathbf{u}} \alpha_{\mathbf{u}} \chi^{\mathbf{u}} \mapsto \min_{\mathbf{u}}(\varphi_F(\mathbf{u})), f \in k[S].$$

Then extend the valuation to $k(S) := \mathrm{Frac}(k[S])$ by defining $\nu_F(\frac{f}{g}) = \nu_F(f) - \nu_F(g)$, for $\frac{f}{g} \in k(S)$. Note that for $\mathbf{u} \in S, \nu_F(\chi^{\mathbf{u}}) = \varphi_F(\mathbf{u})$ and if $\mathbf{u} \in F, \nu_F(\chi^{\mathbf{u}}) = 0$, if $\mathbf{u} \in S \backslash F, \nu_F(\chi^{\mathbf{u}}) > 0$.

Now let $\Sigma = \{S_i\}_i$ be a fan in $\mathbb{Z}[x]^n$. For any $L \in L_\Sigma(1)$, if L has only one element F, then F is a facet of some S_i, and we define $\nu_L = \nu_F$; if L has more than one element, choose one for example F_i and we define $\nu_L = \nu_{F_i}$. In the latter case, for $F_i \sim F_j \in L$, suppose $F_i \preceq S_i, F_j \preceq S_j$ and $S_i + \mathbb{Z}[x](-\mathbf{u}) = S_j + \mathbb{Z}[x](\mathbf{u})$. By the definition of the equivalence relationship $\sim$, there exists a facet F of $S_i + \mathbb{Z}[x](-\mathbf{u})$ such that $F_i^{md} = F_j^{md} = F^{md}$. It follows that F_i, F_j and F have the same standard normal vector and hence $\nu_{F_i} = \nu_{F_j} = \nu_F$. So ν_L is independent of the choice of F_i.

Let X_Σ be the toric $P[\sigma]$-variety associated with the fan Σ and assume that T_Σ is the σ-torus of X_Σ. By Theorem 6.12, each $L \in L_\Sigma(1)$ corresponds to a σ-codimension one irreducible T_Σ-invariant $P[\sigma]$-subvariety $D(L)$ of X_Σ, which is called a *prime divisor* of X_Σ.

Definition 7.2. $\mathrm{Div}(X_\Sigma)$ is the free $\mathbb{Z}[x]$-module generated by the prime divisors of X_Σ as a basis. A *Weil divisor* is an element of $\mathrm{Div}(X_\Sigma)$ which is of the form $\sum_{L \in L_\Sigma(1)} a_L D(L)$.

Let $D = \sum_{L \in L_\Sigma(1)} a_L D(L)$, then D is said to be *effective*, written as $D \geqslant 0$, if $a_L \geqslant 0$ for all L.

Definition 7.3. Let $\Sigma = \{S_i\}_i$ be a fan and X_Σ the toric $P[\sigma]$-variety associated with Σ. Assume $M = \Sigma^{md}$.

1. The *divisor* of $f \in k(M) := \mathrm{Frac}(k[M])$ is defined to be $\mathrm{div}(f) = \sum_L \nu_L(f) D(L)$ where $L \in L_\Sigma(1)$.
2. A divisor of the form $\mathrm{div}(f)$ for some $f \in k(M)$ is called a *principal divisor*, and the set of all principal divisors is denoted by $\mathrm{Div}_0(X_\Sigma)$.
3. For $\mathbf{u} \in M$, $\mathrm{div}(\chi^{\mathbf{u}}) = \sum_{L \in L_\Sigma(1)} \varphi_L(\mathbf{u}) D(L)$ is called a *characteristic divisor*, and the set of all characteristic divisors is denoted by $\mathrm{Div}_c(X_\Sigma)$.
4. Divisors D and E are said to be *linearly equivalent*, written $D \sim E$, if their difference is a principal divisor, i.e. $D - E = \mathrm{div}(f) \in \mathrm{Div}_0(X_\Sigma)$ for some $f \in k(M)$.

If $f, g \in k(M)$, then $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ and $\mathrm{div}(f^a) = a\,\mathrm{div}(f)$, for $a \in \mathbb{Z}[x]$. Thus $\mathrm{Div}_0(X_\Sigma)$ is a $\mathbb{Z}[x]$-submodule of $\mathrm{Div}(X_\Sigma)$. Similarly, $\mathrm{Div}_c(X_\Sigma)$ is also a $\mathbb{Z}[x]$-submodule of $\mathrm{Div}(X_\Sigma)$.

If $D = \sum_i a_i D_i$ is a Weil divisor on X_Σ and $U \subseteq X_\Sigma$ is a nonempty open subset, then $D|_U = \sum_{D_i \cap U \neq \emptyset} a_i D_i \cap U$ is called the *restriction* of D on U.

Definition 7.4. A Weil divisor D on a toric $P[\sigma]$-variety X_Σ is *Cartier* if it is *locally characteristic*, meaning that X_Σ has an open cover $\{U_i\}_{i \in I}$ such that $D|_{U_i}$ is characteristic on U_i for every $i \in I$, namely there exists $\mathbf{u}_i \in M$, such that $D|_{U_i} = \mathrm{div}(\chi^{\mathbf{u}_i})|_{U_i}$ for $i \in I$, and we call $\{(U_i, \mathbf{u}_i)\}_{i \in I}$ the *local data* for D.

We can check that all of the Cartier divisors on X_Σ form a $\mathbb{Z}[x]$-module $\mathrm{CDiv}(X_\Sigma)$ satisfying $\mathrm{Div}_c(X_\Sigma) \subseteq \mathrm{CDiv}(X_\Sigma) \subseteq \mathrm{Div}(X_\Sigma)$.

Definition 7.5. Let X_Σ be the toric $P[\sigma]$-variety of a fan $\Sigma = \{S_i\}_i$. Its *class module* is

$$\mathrm{Cl}(X_\Sigma) := \mathrm{Div}(X_\Sigma)/\mathrm{Div}_c(X_\Sigma)$$

and its *Picard module* is

$$\mathrm{Pic}(X_\Sigma) := \mathrm{CDiv}(X_\Sigma)/\mathrm{Div}_c(X_\Sigma).$$

Obviously, $\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Cl}(X_\Sigma)$ as $\mathbb{Z}[x]$-modules.

Example 7.6. Let $U = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{Z}[x]^n$. Obviously, the toric $P[\sigma]$-variety $\mathscr{X}_U$ defined by U is the σ-affine space $\mathbb{A}^n$. The corresponding affine $P[x]$-semimodule $S = P[x](U) = P[x]^n$ has n facets whose standard normal vectors are $\{\varphi_1, \ldots, \varphi_n\}$ respectively with $\varphi_i(\mathbf{e}_i) = 1, \varphi_i(\mathbf{e}_j) = 0, j \neq i$, $1 \leqslant i, j \leqslant n$. Suppose that the corresponding prime divisors are $\{D_1, \ldots, D_n\}$ respectively. Then for each i, $\mathrm{div}(\chi^{\mathbf{e}_i}) = D_i$. Thus

$$\mathrm{Cl}(\mathbb{A}^n) = \mathbb{Z}[x]D_1 \oplus \cdots \oplus \mathbb{Z}[x]D_n/(D_1, \ldots, D_n) = 0.$$

Example 7.7. Suppose that $U = \{\mathbf{u}_1 = (x,1), \mathbf{u}_2 = (x,2), \mathbf{u}_3 = (x,3)\}$ and $\mathscr{X}_U$ is the affine toric $P[\sigma]$-variety defined by U. The corresponding affine $P[x]$-semimodule $S = P[x](U)$ has two facets $F_1 = P[x](\mathbf{u}_1)$ and $F_2 = P[x](\mathbf{u}_3)$, and we denote their corresponding prime divisors by D_1 and D_2 respectively. The standard normal vector of F_1 is φ_1 with $\varphi_1(\mathbf{u}_1) = 0, \varphi_1(\mathbf{u}_2) = 1, \varphi_1(\mathbf{u}_3) = 2$. The standard normal vector of F_2 is φ_2 with $\varphi_2(\mathbf{u}_1) = 2, \varphi_1(\mathbf{u}_2) = 1, \varphi_1(\mathbf{u}_3) = 0$. So $\mathrm{div}(\chi^{\mathbf{u}_1}) = 2D_2$, $\mathrm{div}(\chi^{\mathbf{u}_2}) = D_1 + D_2$ and $\mathrm{div}(\chi^{\mathbf{u}_3}) = 2D_1$. Thus

$$\mathrm{Cl}(\mathscr{X}_U) = \mathbb{Z}[x]D_1 \oplus \mathbb{Z}[x]D_2 \oplus \mathbb{Z}[x]D_3/(2D_2, D_1 + D_2, 2D_1) \simeq \mathbb{Z}[x]/(2).$$

Example 7.8. Suppose that $U = \{\mathbf{u}_1 = (x,1,1), \mathbf{u}_2 = (1,x,1), \mathbf{u}_3 = (1,1,x), \mathbf{u}_4 = (1,1,1)\}$ and $\mathscr{X}_U$ is the affine toric $P[\sigma]$-variety defined by U. The corresponding affine $P[x]$-semimodule $S = P[x](U)$ has three facets $F_1 = P[x](\{\mathbf{u}_2, \mathbf{u}_3\})$ and $F_2 = P[x](\{\mathbf{u}_1, \mathbf{u}_3\})$, $F_3 = P[x](\{\mathbf{u}_1, \mathbf{u}_2\})$, and we denote their corresponding prime divisors by D_1, D_2 and D_3 respectively. The standard normal vector of F_1 is φ_1 with $\varphi_1(\mathbf{u}_1) = x+2, \varphi_1(\mathbf{u}_2) = \varphi(\mathbf{u}_3) = 0, \varphi_1(\mathbf{u}_4) = 1$. The standard normal vector of F_2 is φ_2 with $\varphi_2(\mathbf{u}_1) = \varphi(\mathbf{u}_3) = 0, \varphi_1(\mathbf{u}_2) = x+2, \varphi_1(\mathbf{u}_4) = 1$. The standard normal vector of F_3 is φ_3 with $\varphi_2(\mathbf{u}_1) = \varphi(\mathbf{u}_2) = 0, \varphi_1(\mathbf{u}_3) = x+2, \varphi_1(\mathbf{u}_4) = 1$. So $\mathrm{div}(\chi^{\mathbf{u}_1}) = (x+2)D_1$, $\mathrm{div}(\chi^{\mathbf{u}_2}) = (x+2)D_2$, $\mathrm{div}(\chi^{\mathbf{u}_3}) = (x+2)D_3$ and $\mathrm{div}(\chi^{\mathbf{u}_4}) = D_1 + D_2 + D_3$. Thus

$$\begin{aligned}\mathrm{Cl}(\mathscr{X}_U) &= \bigoplus_{i=1}^{4} \mathbb{Z}[x]D_i/((x+2)D_1, (x+2)D_2, (x+2)D_3, D_1 + D_2 + D_3)\\ &\simeq \mathbb{Z}[x]/(x+2) \oplus \mathbb{Z}[x]/(x+2).\end{aligned}$$

Example 7.9. The σ-projective space $\mathbb{P}^2$ is defined by $U = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{Z}[x]^2$. The affine $P[\sigma]$-semimodules associated with $\mathbb{P}^2$ are $S_1 = P[x](\{\mathbf{e}_1, \mathbf{e}_2\})$, $S_2 = P[x](\{-\mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_1\})$ and $S_3 = P[x](\{-\mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\})$. S_1 has facets $F_1 = P[x](\mathbf{e}_1)$ and $F_2 = P[x](\mathbf{e}_2)$. S_2 has facets $F_3 = P[x](-\mathbf{e}_1)$ and $F_4 = P[x](\mathbf{e}_2 - \mathbf{e}_1)$. S_3 has facets $F_5 = P[x](-\mathbf{e}_2)$ and $F_6 = P[x](\mathbf{e}_1 - \mathbf{e}_2)$. $F_1 \sim F_3$ has the standard normal vector φ_1 with $\varphi_1(\mathbf{e}_1) = 0, \varphi_1(\mathbf{e}_2) = 1$. $F_2 \sim F_5$ has the standard normal vector φ_2 with $\varphi_2(\mathbf{e}_1) = 1, \varphi_2(\mathbf{e}_2) = 0$. $F_4 \sim F_6$ has the standard normal vector φ_3 with $\varphi_3(\mathbf{e}_1) = -1, \varphi_3(\mathbf{e}_2) = -1$. Denote the corresponding prime divisors by D_1, D_2, D_3 respectively. Then $\mathrm{div}(\chi^{\mathbf{e}_1}) = D_2 - D_3$, $\mathrm{div}(\chi^{\mathbf{e}_2}) = D_1 - D_3$. Thus

$$\mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}[x]D_1 \oplus \mathbb{Z}[x]D_2 \oplus \mathbb{Z}[x]D_3/(D_2 - D_3, D_1 - D_3) \simeq \mathbb{Z}[x].$$

In the same way, we can show that $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}[x], n \geqslant 1$.

Proposition 7.10. Let $X = \mathrm{Spec}^{P[\sigma]}(k[S])$ be an affine toric $P[\sigma]$-variety. Then

(a) Every Cartier divisor on X is a characteristic divisor;

(b) $\mathrm{Pic}(X) = 0$.

Proof. (a) Let H be the intersection of all faces of S which is still a face of S. Then $D(H) \subseteq \bigcap_{F \in F_S(1)} D(F)$. Fix a point $p \in D(H)$. Since D is Cartier, it is locally characteristic, and in particular is characteristic in a neighbourhood U of p, i.e. $D|_U = \mathrm{div}(\chi^{\mathbf{u}})|_U$ for some $\mathbf{u} \in S^{md}$. Since $p \in U \cap D(F)$ for all $F \in F_S(1)$, we have $D = \mathrm{div}(\chi^{\mathbf{u}})$.

(b) It follows from (a). □

An affine $P[x]$-semimodule S is said to be *compact* if $\bigcap_{F \in F_S(1)} F^{md} = \{\mathbf{0}\}$.

Proposition 7.11. Let X_Σ be the toric $P[\sigma]$-variety of a fan Σ and $M = \Sigma^{md}$. If Σ contains a compact affine $P[x]$-semimodule, then $\mathrm{Pic}(X_\Sigma)$ is a $\mathbb{Z}[x]$-lattice.

Proof. Because of Lemma 2.7, it suffices to show that if D is a Cartier divisor and gD is the divisor of a character for some $g \in P[x]^*$, then the same is true for D. Write $D = \sum_L a_L D(L)$ and assume that $gD = \mathrm{div}(\chi^{\mathbf{u}}), \mathbf{u} \in M$. Let S be a compact affine $P[x]$-semimodule in Σ. Since D is Cartier, its restriction to $X_S := \mathrm{Spec}^{P[\sigma]}(k[S])$ is also Cartier. Assume that $D|_{X_S} = \sum_{F \in F_S(1)} a_F D(F)$. This is

characteristic on X_S by Proposition 7.10, so there exists $\mathbf{u}' \in M$ such that $D|_{X_S} = \mathrm{div}(\chi^{\mathbf{u}'})|_{X_S}$. This implies $a_F = \varphi_F(\mathbf{u}')$, for all $F \in F_S(1)$. On the other hand, $gD = \mathrm{div}(\chi^{\mathbf{u}})$ implies $ga_L = \varphi_L(\mathbf{u})$, for all $L \in L_\Sigma(1)$. It follows $\varphi_F(g\mathbf{u}') = ga_F = \varphi_F(\mathbf{u})$, for all $F \in F_S(1)$, i.e. $\varphi_F(g\mathbf{u}' - \mathbf{u}) = 0$ for all $F \in F_S(1)$. So there exists $g_F \in P[x]^*$ such that $g_F(g\mathbf{u}' - \mathbf{u}) \in F^{md}$, for all $F \in F_S(1)$. Thus $(\prod_{F\in F_S(1)} g_F)(g\mathbf{u}'-\mathbf{u}) \in \bigcap_{F\in F_S(1)} F^{md}$. Since S is compact, $\bigcap_{F\in F_S(1)} F^{md} = \{\mathbf{0}\}$ and hence $g\mathbf{u}'-\mathbf{u} = \mathbf{0}$. It follows $D = \mathrm{div}(\chi^{\mathbf{u}'})$. □

Definition 7.12. A toric $P[\sigma]$-variety X is *smooth* if $\mathrm{Cl}(X) = \mathrm{Pic}(X)$.

Definition 7.13. Suppose that S is an affine $P[x]$-semimodule and $M = S^{md}$. For a facet F of S, assume φ_F is the standard normal vector of F. If $\{\varphi_F \mid F \in F_S(1)\}$ forms a basis of the free module $M^* = \mathrm{Hom}_{\mathbb{Z}[x]}(M, \mathbb{Z}[x])$, then we say that S is *smooth*. Let Σ be a fan. If for every $S \in \Sigma$, S is smooth, then we say that Σ is *smooth*.

In the algebraic case, the smoothness of a toric variety is equivalent to the smoothness of the corresponding fan. We will generalize this result to the difference case. Firstly, let us prove some lemmas.

Lemma 7.14. *Let X_Σ be the toric $P[\sigma]$-variety of a fan Σ and $Z = Spec^{P[\sigma]}(k[S])$ for some $S \in \Sigma$. Let $D_1, \ldots, D_s$ be the irreducible components of $X_\Sigma \backslash Z$ that are prime divisors. Then the sequence*

$$\sum_{j=1}^{s} \mathbb{Z}[x]D_j \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow \mathrm{Cl}(Z) \longrightarrow 0$$

is exact, where the first map sends $\sum_{j=1}^{s} a_j D_j$ to its divisor class in $\mathrm{Cl}(X_\Sigma)$ and the second is induced by the restriction to Z.

Proof. Let $D' = \sum_i a_i D'_i \in \mathrm{Cl}(Z)$ with D'_i a prime divisor on Z. Then the closure $\overline{D'_i}$ of D'_i in X_Σ is a prime divisor of X_Σ and $D = \sum_i a_i \overline{D'_i}$ satisfies $D|_Z = D'$. Hence $\mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(Z)$ is surjective.

Since each D_j restricts to 0 in $\mathrm{Div}(Z)$, the composition of the two maps is trivial. To prove the exactness, suppose that $[D] \in \mathrm{Cl}(X_\Sigma)$ restricts to 0 in $\mathrm{Cl}(Z)$. This means that $D|_Z$ is the divisor of some $\chi^{\mathbf{u}} \in k[S^{md}] = k[\Sigma^{md}]$, i.e. $D|_Z = \mathrm{div}(\chi^{\mathbf{u}})|_Z$. This implies that $D - \mathrm{div}(\chi^{\mathbf{u}})$ is supported on $X\backslash Z$, i.e. $D - \mathrm{div}(\chi^{\mathbf{u}}) \in \sum_{j=1}^{s} \mathbb{Z}[x]D_j$. So $[D] \in [\sum_{j=1}^{s} \mathbb{Z}[x]D_j]$ as desired. □

Lemma 7.15. *Let M be a free $\mathbb{Z}[x]$-lattice and $M^* = \mathrm{Hom}_{\mathbb{Z}[x]}(M, \mathbb{Z}[x])$ its dual $\mathbb{Z}[x]$-lattice. For a subset $\{\varphi_1, \ldots, \varphi_s\} \subseteq M^*$, define a map*

$$\Phi\colon \mathbb{Z}[x]^s \longrightarrow M^*, (a_1, \ldots, a_s) \mapsto \sum_{i=1}^{s} a_i\varphi_i.$$

The dual map of Φ is

$$\Phi^*\colon \mathrm{Hom}_{\mathbb{Z}[x]}(M^*, \mathbb{Z}[x]) \simeq M \longrightarrow \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]^s, \mathbb{Z}[x]) \simeq \mathbb{Z}[x]^s$$

with $\Phi^*(\mathbf{m})\colon \mathbb{Z}[x]^s \to \mathbb{Z}[x], \mathbf{a} \mapsto \Phi(\mathbf{a})(\mathbf{m})$, for every $\mathbf{m} \in M$. Then Φ is an isomorphism if and only if Φ^* is an isomorphism.

Proof. Suppose Φ^* is an isomorphism. For any $\mathbf{a} = (a_1, \ldots, a_s) \in \ker(\Phi)$, and $\mathbf{m} \in M$, $\Phi^*(\mathbf{m})(\mathbf{a}) = \Phi(\mathbf{a})(\mathbf{m}) = 0$, i.e. $\Phi^*(M)(\mathbf{a}) = 0$. Since Φ^* is surjective, we have $\Phi^*(M) = \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]^s, \mathbb{Z}[x])$. So $\mathbf{a} = \mathbf{0}$. Thus $\ker(\Phi) = \mathbf{0}$ and Φ is injective. Since Φ^* is an isomorphism, $\mathrm{rank}(M^*) = \mathrm{rank}(M) = s$. Therefore, $\{\varphi_1, \ldots, \varphi_s\}$ is linearly independent and generates $M^* \otimes \mathbb{Q}(x)$ as a basis. To prove that Φ is also surjective, we only need to show that $\Phi(\mathbb{Z}[x]^s)$ is $\mathbb{Z}[x]$-saturated, i.e. for any $g \in \mathbb{Z}[x]^*$ and $\varphi \in M^*$, if $g\varphi \in \Phi(\mathbb{Z}[x]^s)$, then $\varphi \in \Phi(\mathbb{Z}[x]^s)$. Assume $\mathbf{a} = (a_1, \ldots, a_s)$ and $\Phi(\mathbf{a}) = g\varphi$. Let $\{\mathbf{e}_i\}_{i=1}^s$ be the standard basis of $\mathbb{Z}[x]^s$. Since Φ^* is surjective, then for every i, there exists an $\mathbf{m}_i \in M$ such that $\Phi^*(\mathbf{m}_i)(\mathbf{e}_i) = 1, \Phi^*(\mathbf{m}_i)(\mathbf{e}_j) = 0, j \neq i$. Then $g\varphi(\mathbf{m}_i) = \Phi(\mathbf{a})(\mathbf{m}_i) = \sum_{i=1}^{s} \mathbf{a}_i\varphi(\mathbf{m}_i) = \mathbf{a}_i$. Let $\mathbf{a}' = (\varphi(\mathbf{m}_1), \ldots, \varphi(\mathbf{m}_s))$. Then $g\Phi(\mathbf{a}') = \Phi(\mathbf{a}) = g\varphi$ and $\Phi(\mathbf{a}') = \varphi$. Hence Φ is an isomorphism.

Since $\Phi^{**} = \Phi$, the converse follows easily. □

Theorem 7.16. *Let X_Σ be the toric $P[\sigma]$-variety of a fan Σ. Assume $M = \Sigma^{md}$ is a free $\mathbb{Z}[x]$-module. Then X_Σ is smooth if and only if Σ is smooth.*

Proof. By definition, X_Σ is smooth if and only if $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ which is equivalent to the fact that every Weil divisor on X_Σ is Cartier. For the necessity, assume that every Weil divisor on X_Σ is Cartier. By Lemma 7.14, for any $S \in \Sigma$, every Weil divisor on $X_S := \mathrm{Spec}^{P[\sigma]}(k[S])$ is Cartier. Then by Proposition 7.10, $\mathrm{Cl}(X_S) = \mathrm{Pic}(X_S) = 0$. Since we have the following exact sequence:

$$0 \longrightarrow M \xrightarrow{\theta} \mathrm{Div}(X_S) \longrightarrow \mathrm{Cl}(X_S) \longrightarrow 0,$$

where θ maps $\mathbf{u} \in M$ to the divisor of $\chi^{\mathbf{u}}$, we know that θ is an isomorphism. If $\{\varphi_F \mid F \in F_S(1)\} = \{\varphi_1, \ldots, \varphi_s\}$, then this map becomes

$$\theta\colon M \longrightarrow \mathbb{Z}[x]^s, \mathbf{m} \mapsto (\varphi_1(\mathbf{m}), \ldots, \varphi_s(\mathbf{m})), \forall \mathbf{m} \in M. \tag{7.1}$$

Now define $\Phi\colon \mathbb{Z}[x]^s \to M^*$ by $\Phi(a_1, \ldots, a_s) = \sum_{i=1}^s a_i\varphi_i$. The dual map

$$\Phi^*\colon M \simeq \mathrm{Hom}_{\mathbb{Z}[x]}(M^*, Z[x]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]^s, \mathbb{Z}[x]) \simeq \mathbb{Z}[x]^s$$

is easily seen to be (7.1). Since Φ^* is an isomorphism, then Φ is an isomorphism by Lemma 7.15. The injectivity of Φ implies that $\{\varphi_1, \ldots, \varphi_s\}$ is linearly independent. The surjectivity of Φ implies that $\{\varphi_1, \ldots, \varphi_s\}$ generates M^* as a $\mathbb{Z}[x]$-module. So $\{\varphi_1, \ldots, \varphi_s\}$ is a basis of M^* and hence S is smooth. So Σ is smooth.

Every step in the above proof is invertible, so the sufficiency follows. □

Example 7.17. The σ-projective space $\mathbb{P}^n$ is defined by $U = \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where $\{\mathbf{e}_i\}_{i=1}^n$ is the standard basis of $\mathbb{Z}[x]^n$. Let $S_0 = P[x](\{\mathbf{e}_1, \ldots, \mathbf{e}_n\})$ and $S_i = P[x](U - \mathbf{e}_i), i = 1, \ldots, n$. The fan associated with $\mathbb{P}^n$ is $\{S_i\}_{i=0}^n$ by Example 6.7. It is easy to check that $S_0 = P[x](\{\mathbf{e}_1, \ldots, \mathbf{e}_n\})$ and $S_i = P[x](U - \mathbf{e}_i), i = 1, \ldots, n$ are smooth. So $\mathbb{P}^n$ is smooth by Theorem 7.16 and $\mathrm{Pic}(\mathbb{P}^n) = \mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}[x]$.

8 Conclusions

In this paper, we first introduce the concept of $P[\sigma]$-varieties and initiate the study of toric $P[\sigma]$-varieties. We define affine toric $P[\sigma]$-varieties and establish the connections between affine toric $P[\sigma]$-varieties and affine $P[x]$-semimodules. We show that the category of affine toric $P[\sigma]$-varieties with toric morphisms is antiequivalent to the category of affine $P[x]$-semimodules with $P[x]$-semimodule morphisms. Moreover, we show that there is a one-to-one correspondence between the irreducible T-invariant $P[\sigma]$-subvarieties of an affine toric $P[\sigma]$-variety X and the faces of the corresponding affine $P[x]$-semimodule, where T is the σ-torus of X. Besides, there is also a one-to-one correspondence between the T-orbits of the affine toric σ-variety X and the faces of the corresponding affine $P[x]$-semimodule.

We also define projective toric $P[\sigma]$-varieties in a σ-projective space and define abstract toric $P[\sigma]$-varieties associated with a fan by gluing affine toric $P[\sigma]$-varieties. It turns out that both affine toric $P[\sigma]$-varieties and projective toric $P[\sigma]$-varieties are abstract toric $P[\sigma]$-varieties. The irreducible invariant $P[\sigma]$-subvarieties-faces correspondence is generalized to abstract toric $P[\sigma]$-varieties. By virtue of the correspondence, we can develop a divisor theory on abstract toric $P[\sigma]$-varieties and establish the connections between the properties of a toric $P[\sigma]$-variety and its divisor class modules.

The divisor theory for toric $P[\sigma]$-varieties developed in this paper is not complete. In algebraic geometry, many applications of the divisor theory on algebraic varieties, in particular on toric varieties, are revealed. We hope that we can give more applications of the divisor theory on toric $P[\sigma]$-varieties in the future work.

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