Positive Zeros of Systems of Polynomial Equations

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Bounds of numbers of (real) positive zeros

Problem

Given a (parametric) polynomial or a system of (parametric) polynomials, bound the number of its positive (real) zeros from above and from below. More specifically, let $\mathscr{A} \in \mathbb{N}^{m\times n}$ (or $\mathbb{R}^{m\times n}$) and $C \in \mathbb{R}^{n\times m}$, bound the number of positive (real) zeros of $C\mathrm{x}^{\mathscr{A}}$ from above and from below in terms of $\mathscr{A}$ and C.

- Polynomial optimization
- Chemical reaction networks
- Algebraic statistics
- $\cdots$

Bounds of numbers of positive zeros

For given $\mathscr{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\} \subseteq \mathbb{N}^n$ (or $\mathbb{R}^n$), let

$$M_{\mathscr{A}} := \{\mathrm{x}^{\mathscr{A}} = (\mathrm{x}^{\boldsymbol{\alpha}_1}, \ldots, \mathrm{x}^{\boldsymbol{\alpha}_m}) \mid \mathrm{x} \in \mathbb{R}_+^n\}.$$

Then to study positive zeros of $C\mathrm{x}^{\mathscr{A}}$ is essentially to study the intersection of the non-convex set $M_{\mathscr{A}}$ and an affine plane $C\mathrm{y} = 0$, which turns out to be hard.

Descartes' rule of signs

For a univariate polynomial $f(x) = \sum_{i=1}^{m} c_i x^{\alpha_i}$ with $\alpha_1 < \ldots < \alpha_m$, define $\mathrm{svar}(f) = \mathrm{svar}(C) := \#\{i : c_i c_{i+1} < 0\}$, the number of sign variations between consecutive c_i.

Descartes' rule of signs

Given a univariate real polynomial $f(x)$, let $N(f)$ be the number of positive zeros of f (counted with multiplicity). Then $N(f) \leq \mathrm{svar}(f)$. Additionally, $\mathrm{svar}(f) - N(f)$ is an even number.

Various extensions of Descartes' rule of signs

- Extensions to other univariate analytic functions: exponential functions, trigonometric functions and orthogonal polynomials (Dimitrov and Rafaeli, 2009)
- Exact version of Descartes' rule of signs (Avendaño, 2010)
- Extension to tropical algebra (Forsgård, Novikov and Shapiro, 2017)
- Extension to matrix polynomials (Cameron and Psarrakos, 2019)
- ......

Problem 1

Problem 1: Give sufficient conditions such that Descartes' rule of signs holds exactly.

More specifically, fixing the exponents $\mathscr{A}$ and signs of the coefficients $\mathrm{sgn}(C)$, for which C, the number of positive zeros of $C\mathrm{x}^{\mathscr{A}}$ attains the maximal value $\mathrm{svar}(C)$?

Totally open.

Problem 2

Theorem (Poincaré, 1888)

For a non-zero $f \in \mathbb{R}[x]$, *there exists* $g \in \mathbb{R}[x]$ *such that* $N(f) = \mathrm{svar}(gf)$.

Remark: g can be chosen as $(x+1)^k$ (Avendaño, 2010).

Problem 2: Give a degree bound for g in the above theorem. (Powers and Reznick, 2001)

Theorem (Riggs, 2020)

For a non-zero $f \in \mathbb{R}[x]$, *there exists* $g \in \mathbb{R}[x]$ *such that* $N(f) = \mathrm{svar}(gf)$ *and* $\deg(g) \le (N(f)+1)\sum_{i=1}^{m}(\lceil \frac{\pi}{\arg(\alpha_i)} \rceil - 2) + N(f)(\deg(f) - N(f))$, *where* $\alpha_1, \ldots, \alpha_m$ *are all the non-real roots of* f *with positive imaginary part (multiple roots are repeated).*

Problem 3

Problem 3: Give lower bounds of positive zeros for a univariate polynomial.

Given $\mathscr{A} \in \mathbb{N}^m$ and $C \in \mathbb{R}^m$, let $n_{\mathscr{A}}(C)$ be the number of positive zeros of $C\mathrm{x}^{\mathscr{A}}$ (counted with multiplicity). Then

$$0 \text{ or } 1 \leq n_{\mathscr{A}}(C) \leq \mathrm{svar}(C).$$

Can we give a better lower bound for $n_{\mathscr{A}}(C)$?

Problem 4

For $\mathscr{A} = \{0, 1, \ldots, m\}$ and a sign pattern of coefficients $\xi \in \{+, -\}^{m+1}$, let ξ^+, ξ^- be the upper bounds of numbers of positive zeros and negative zeros provided by Descartes' rule of signs respectively. Any triple (ξ, η^+, η^-) is called an admissible set if $\eta^+ \leq \xi^+, \eta^- \leq \xi^-$ and $\xi^+ - \eta^+, \xi^- - \eta^-$ are both even.

An admissible set (ξ, η^+, η^-) is realizable if there exists a polynomial with sign pattern ξ and having η^+ positive zeros, η^- negative zeros.

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Problem 4: Which admissible sets are realizable?
Known for polynomials of degree ≤ 8 (Kostov and Shapiro, 2020).
No conjecture. Widely open.

Problem 5

Problem 5: Give a multivariate version of Descartes' rule of signs.

Generally no answer. No conjecture.

Upper bounds in the multivariate case

For each pair $l, n \in \mathbb{N}$, define the **Khovanskii number** $X(l, n)$ to be the maximum number of nondegenerate positive zeros to a system of n polynomials in n variables with $1 + l + n$ monomials.

Upper bounds in the multivariate case

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Theorem (Khovanskii, 1980)

A system of n polynomials in n variables with a total of $1 + l + n$ distinct monomials has at most

$$2^{\binom{l+n}{2}}(n+1)^{l+n}$$

nondegenerate positive zeros.

Khovanskii's Theorem gives an upper bound on $X(l, n)$, but that bound is enormous. For example, when $l = n = 2$, the bound is 5184.

Upper bounds in the multivariate case

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Upper bounds in the multivariate case

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Theorem (Li, Rojas and Wang, 2003)

Two trinomials in two variables have at most five nondegenerate positive zeros.

This bound is sharp as the following example by Haas shows.

Upper bounds in the multivariate case

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Example (Haas, 2002)

The following system in x and y

$$\begin{cases} x^6 + \frac{78}{55}y^3 - y = 0 \\ y^6 + \frac{78}{55}x^3 - x = 0 \end{cases}$$

has five positive zeros.

Upper bounds in the multivariate case

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Theorem (Bihan and Sottile, 2007)

$$X(l, n) < \frac{e^2 + 3}{4} 2^{\binom{l}{2}} n^l.$$

Upper bounds in the multivariate case

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Upper bounds in the multivariate case

The best results we know on $X(l, n)$ are:

- $X(0, n) = 1$
- $X(1, n) = n + 1$ (Bihan, 2007)
- $7 \leq X(2, 2) \leq 15$ (Hilany, 2018)

A partial generalization of Descartes' rule of signs

Suppose $\mathscr{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n+2}\} \subseteq \mathbb{N}^n$ is a circuit. There exists a unique (up to a scalar) $\lambda = (\lambda_i) \in (\mathbb{R}^*)^{n+2}$ s.t. $\sum_{i=1}^{n+2} \lambda_i \boldsymbol{\alpha}_i = 0$.

Theorem (Bihan and Dickenstein, 2017)

Suppose $\mathscr{A} \subseteq \mathbb{N}^n$ is a circuit with $\lambda \in (\mathbb{R}^*)^{n+2}$ and C is uniform. Let $n_{\mathscr{A}}(C)$ be the number of positive zeros of $C\mathrm{x}^{\mathscr{A}}$ which assumes to be finite. There exists a specific permutation τ of λ determined by C so that

$$n_{\mathscr{A}}(C) \leq \mathrm{svar}(\tau(\lambda)).$$

Remark: In a similar manner, a refined bound was provided by Bihan, Dickenstein, and Forsgård in 2020, which is sharp for given $\mathscr{A}$.

A partial generalization of Descartes' rule of signs

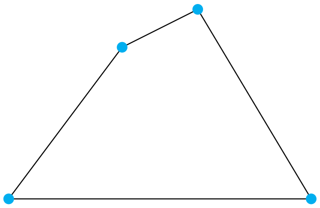
Suppose $\mathscr{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n+2}\} \subseteq \mathbb{N}^n$ is a circuit with $\lambda = (\lambda_i) \in (\mathbb{R}^*)^{n+2}$. Let $s_+ := \#\{i : \lambda_i > 0\}$, $s_- := \#\{i : \lambda_i < 0\}$ and $\sigma(\mathscr{A}) := \min\{s_+, s_-\}$.

Theorem (Bihan and Dickenstein, 2017)

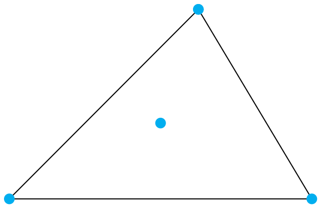
Suppose $\mathscr{A} \subseteq \mathbb{N}^n$ is a circuit with signature (s_+, s_-). Let $n_{\mathscr{A}}(C)$ be the number of positive zeros of $C\mathbf{x}^{\mathscr{A}}$ which assumes to be finite. Then

$$n_{\mathscr{A}}(C) \le \begin{cases} 2\sigma(\mathscr{A}), & \textit{if } s_+ \neq s_-, \\ 2\sigma(\mathscr{A}) - 1, & \textit{if } s_+ = s_-. \end{cases}$$

When $n = 2$



$n_{\mathscr{A}}(C) \leq 3$



$n_{\mathscr{A}}(C) \leq 2$

Lower bounds of numbers of real zeros

Consider a system of real polynomial equations F with support $\mathscr{A} \subseteq \mathbb{N}^n$.
$X_{\mathscr{A}}$: the toric variety defined by $\mathscr{A}$
Construct a map

$$\pi : X_{\mathscr{A}} \longrightarrow \mathbb{P}^n$$

Restrict to the real part for $Y_{\mathscr{A}} := X_{\mathscr{A}} \cap \mathbb{RP}^{|\mathscr{A}|}$

$$g : Y_{\mathscr{A}} \longrightarrow \mathbb{RP}^n$$

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Theorem (Sottile, 2011)

The absolute value of the mapping degree $|\mathrm{mdeg}(g)|$ is a lower bound on the number of real zeros of F.

Existence of positive zeros

Existence of positive zeros (case I)

$\mathrm{conv}(\mathscr{A})$: the convex hull of a finite set $\mathscr{A} \subseteq \mathbb{N}^n$
$V(\Delta)$: the vertex set of a polytope Δ

Theorem (Wang, 2019)

Let F be the following system of polynomial equations

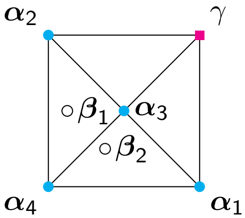
$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathrm{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathrm{x}^{\boldsymbol{\beta}} = 0,$$

where $\mathscr{A} \subseteq \mathbb{N}^n$, $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ and $\boldsymbol{\gamma} \in V(\Delta)$, $\mathscr{B} \subseteq \Delta^\circ \cap \mathbb{N}^n$ with $\Delta = \mathrm{conv}(\mathscr{A} \cup \{\boldsymbol{\gamma}\})$. Assume that $\dim(\Delta) = n$, Δ is simple at $\boldsymbol{\gamma}$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathrm{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathrm{x}^{\boldsymbol{\beta}}$ is not nonnegative over $\mathbb{R}^n_+$. Then F has at least one positive zero.

An example

The following system of polynomial equations with $\mathscr{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\} = \{(8,0),(0,8),(4,4),(0,0)\}$, $\mathscr{B} = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2\} = \{(1,4),(3,2)\}$ and $\gamma = (8,8)$ has at least one positive zero.

$$\begin{cases} -8y^8 - 4x^4y^4 - 8 + 21xy^4 + 5x^3y^2 = 0 \\ -8x^8 - 4x^4y^4 - 8 + 12xy^4 + 6x^3y^2 = 0 \end{cases}$$



Existence of positive zeros (case II)

Theorem (Wang, 2019)

Let F be the following system of polynomial equations

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\gamma)\mathrm{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\gamma)\mathrm{x}^{\boldsymbol{\beta}} = 0,$$

where $\mathscr{A}\cup\{\gamma\}\subseteq\mathbb{N}^n$, $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ and $\mathscr{B}\subseteq\Delta^\circ\cap\mathbb{N}^n$ with $\Delta = \mathrm{conv}(\mathscr{A}\cup\{\gamma\})$. Assume that $\dim(\Delta) = n$, Δ is simple at some vertex $\boldsymbol{\alpha}_0\in\mathscr{A}$ ($\boldsymbol{\alpha}_0\neq\gamma$) and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathrm{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathrm{x}^{\boldsymbol{\beta}}$ is not nonnegative over $\mathbb{R}^n_+$. Then F has at least one positive zero.

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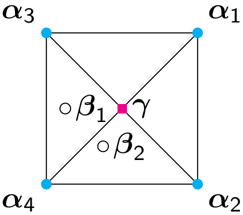
$$\begin{cases} 4x^8y^8 + 4x^8 - 4y^8 - 4 + 9xy^4 + x^3y^2 = 0 \\ 4x^8y^8 - 4x^8 + 4y^8 - 4 + 2x^3y^2 = 0 \end{cases}$$



Existence of positive zeros (case III)

Theorem (Wang, 2019)

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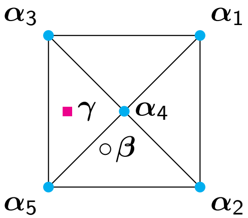
where $\mathscr{A} \subseteq \mathbb{N}^n$, $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ and $\mathscr{B} \cup \{\gamma\} \subseteq \Delta^{\circ} \cap \mathbb{N}^n$ with $\Delta = \mathrm{conv}(\mathscr{A})$. Assume that $\dim(\Delta) = n$, Δ is simple at some vertex and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathrm{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathrm{x}^{\boldsymbol{\beta}}$ is nonnegative over $\mathbb{R}^n_+$. Then F has at least one positive zero.

An example

The following system of polynomial equations with $\mathscr{A} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5\} = \{(8,8),(8,0),(0,8),(4,4),(0,0)\}$, $\mathscr{B} = \{\boldsymbol{\beta}\} = \{(3,2)\}$ and $\gamma = (1,4)$ has a positive zero.

$$\begin{cases} 7x^8y^8 + 7x^8 - y^8 + 3x^4y^4 - 1 - 2x^3y^2 = 0 \\ 4x^8y^8 - 4x^8 + 4y^8 - 4 + 2x^3y^2 = 0 \end{cases}$$



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Existence of global minimizers

$\mathrm{New}(f)$: the Newton polytope of a polynomial f

Theorem (Wang, 2019)

Suppose $f = \sum_{\alpha \in \mathscr{A}} c_{\alpha} \mathrm{x}^{\alpha} - \sum_{\beta \in \mathscr{B}} d_{\beta} \mathrm{x}^{\beta} \in \mathbb{R}[\mathrm{x}]$, $c_{\alpha}, d_{\beta} > 0$ such that $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B} \subseteq \mathrm{New}(f)^{\circ} \cap \mathbb{N}^n$, $\dim(\mathrm{New}(f)) = n$. Assume that $\mathrm{conv}(\mathscr{A} \cup \{0\})$ is simple at 0. If 0 is not a global minimizer of f, then f has a global minimizer in $\mathbb{R}^n_+$.

Remark: Generally deciding the existence of global minimizers of a multivariate polynomial is NP-hard.

Main literature

- S. Müller, E. Feliu, et al., *Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry*, 2016.
- F. Bihan and A. Dickenstein, *Descartes' Rule of Signs for Polynomial Systems supported on Circuits*, 2017.
- J. Wang, *Systems of Polynomials with At Least One Positive Real Zero*, 2019.
- F. Bihan, A. Dickenstein and M. Giaroli, *Sign conditions for the existence of at least one positive solution of a sparse polynomial system*, 2020.
- F. Bihan, A. Dickenstein and J. Forsgård, *Optimal Descartes' Rule of Signs for Circuits*, 2020.

Thanks for your attention!