Nonnegativity, Sparsity and Polynomial Optimization

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Joint work with Victor Magron and Jean B. Lasserre

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21 April, 2022

Outline

1. Background on polynomial nonnegativity
   - SOS decomposition
   - SONC decomposition
2. Sparsity in the moment-SOS hierarchy
   - Correlative sparsity (CS)
   - Term sparsity (TS)
3. SONC decomposition and second order cone representation
4. Numerical experiments

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Certify polynomial nonnegativity

Problem

Given a multivariate polynomial f, decide if it is nonnegative and if so, provide a certificate of nonnegativity.

- A central problem in real algebraic geometry
- Widely appear in numerous fields
- Closely related to polynomial optimization
- NP-hard in general

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SOS decomposition

- SOS (sum of squares) decomposition:

$$f = f_1^2 + \cdots + f_t^2 \quad \rightsquigarrow \quad f \text{ is nonnegative}$$

Example: $f = 1 + 2x + 2x^2 + 2xy + y^2 = (1 + x)^2 + (x + y)^2$

- Hilbert, 1888:

"nonnegative polynomials = SOS" $\Leftrightarrow n = 1;\ d = 2;\ n = 2, d = 4$

- Artin, 1927: "nonnegative polynomials = rational SOS"
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Gram matrix and semidefinite programming

- f: $2d$, $v_d = [1, x_1, \ldots, x_n, x_1^d, \ldots, x_n^d]$
- f admits an SOS decomposition $\iff$ there exists a PSD matrix G s.t.

$$f = v_d \cdot G \cdot v_d^{\intercal} \quad \rightsquigarrow \quad \text{SDP}$$

- G is called a Gram matrix of f, which is of size $\binom{n+d}{n}$

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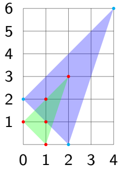
Structural SOS decomposition

- Newton polytope: $f = \sum f_i^2 \Longrightarrow \mathrm{New}(f_i) \subseteq \frac{1}{2}\mathrm{New}(f)$



$f = 4x_1^4x_2^6 + x_1^2 - x_1x_2^2 + x_2^2$

- correlative sparsity

$$f(\mathbf{x}) = f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) \in \mathrm{SOS} \rightsquigarrow f_1(\mathbf{x}_1) \in \mathrm{SOS}, f_2(\mathbf{x}_2) \in \mathrm{SOS}$$

- term sparsity

$$\mathbf{x}^{\beta} \cdot \mathbf{x}^{\gamma} \notin \mathrm{supp}(f), \beta + \gamma \notin (2\mathbb{N})^n \quad \rightsquigarrow \quad G_{\beta\gamma} = 0$$

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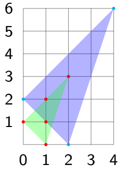

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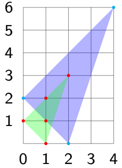

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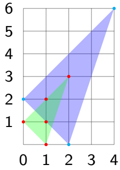

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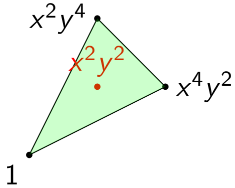
SONC (sum of nonnegative circuits) decomposition

- $M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ (arithmetic-geometric mean inequality $\Rightarrow$ nonnegativity)



- circuit polynomial: $f = \sum_{\alpha \in \mathscr{A}} c_\alpha \mathbf{x}^\alpha - d_\beta \mathbf{x}^\beta$, $\alpha \in (2\mathbb{N})^n$, $c_\alpha > 0$, $\mathscr{A}$ the vertex set of a simplex, $\beta \in \mathrm{conv}(\mathscr{A})^\circ$
- SONC decomposition: $f = f_1 + \cdots + f_t$ with each f_i being a nonnegative circuit polynomial

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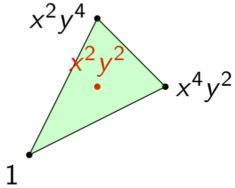

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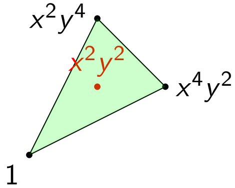

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Sufficient conditions for SONC decompositions

Theorem (Wang, 2022)

Suppose that f is a nonnegative polynomial with exactly one negative term. Then f admits a SONC decomposition.

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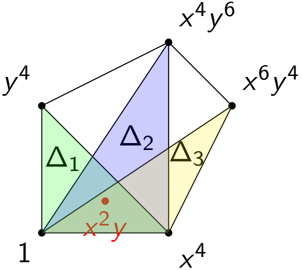

SONC decompositions preserve term sparsity

Theorem (Wang, 2022)

Suppose f is a SONC polynomial. Then f admits a SONC decomposition:

$$f = \sum_{\mathrm{supp}(f_i)\subseteq\mathrm{supp}(f)} f_i,$$

where each f_i is a nonnegative circuit polynomial. Moreover, we can further assume that there is no cancellation occurring in the above decomposition.

Sparsity in the moment-SOS hierarchy

Polynomial optimization

- Polynomial optimization problem:

$$f^* := \begin{cases} \inf & f \\ \text{s.t.} & g_j \geq 0, \quad j = 1, \ldots, m \\ & (h_i = 0, \quad i = 1, \ldots, m') \end{cases}$$

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The goals

- Compute the global optimal value
- Certify global optimality
- Extract global optimal solutions
- Approximate the global optimal value if the exact computation is expensive/impossible/unnecessary

The moment-SOS hierarchy (also known as Lasserre's hierarchy) is a well-established tool to handle POPs and achieve the above goals.

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Moment matrix and localizing matrix

- $\mathbb{N}_r^n := \{\boldsymbol{\beta} = (\beta_i) \in \mathbb{N}^n \mid \sum_{i=1}^n \beta_i \le r\}$
- r-th order moment matrix $M_r(\mathbf{y})$:

$$[M_r(\mathbf{y})]_{\boldsymbol{\beta}\boldsymbol{\gamma}} := y_{\boldsymbol{\beta}+\boldsymbol{\gamma}}, \quad \forall \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}_r^n$$

- Given $g = \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, r-th order localizing matrix $M_r(g\mathbf{y})$:

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- $\mathbf{x} = x$, $g = 1 - x^2$:

$$M_2(\mathbf{y}) = \begin{matrix} & 1 & x & x^2 \\ 1 & y_0 & y_1 & y_2 \\ x & y_1 & y_2 & y_3 \\ x^2 & y_2 & y_3 & y_4 \end{matrix}, \quad M_1(g\mathbf{y}) = \begin{matrix} & 1 & x \\ 1 & y_0 - y_2 & y_1 - y_3 \\ x & y_1 - y_3 & y_2 - y_4 \end{matrix}$$

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Moment relaxation

- Moment relaxation:

$$\theta_r := \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_r(\mathbf{y}) \succeq 0, \\ & M_{r-d_j}(g_j\mathbf{y}) \succeq 0, \quad j = 1, \ldots, m, \\ & y_{\mathbf{0}} = 1. \end{cases}$$

SOS relaxation

- $S = \{\mathbf{x} \in \mathbb{R}^n \mid g_1 \geq 0, \ldots, g_m \geq 0\}$
- Dual to the polynomial optimization problem:

$$f^* = \sup_{\lambda}\{\lambda : f(\mathbf{x}) - \lambda \geq 0, \forall \mathbf{x} \in S\}$$

- $P_S(\mathbf{x}) := \{g(\mathbf{x}) \in \mathbb{R}[\mathbf{x}] \mid g(\mathbf{x}) \geq 0 \text{ over } S\} \rightsquigarrow$ intractable
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Quadratic module

- $\Sigma(\mathbf{x}) := \{f \in \mathbb{R}[\mathbf{x}] \mid f = \sum_i f_i^2, f_i \in \mathbb{R}[\mathbf{x}]\}$
- Quadratic module: Given $\mathbf{g} = \{g_j\}_{j=1}^m \subseteq \mathbb{R}[\mathbf{x}]$,

$$\mathcal{Q}(\mathbf{g}) := \left\{\sigma_0 + \sum_{j=1}^{m} \sigma_j g_j \mid \sigma_j \in \Sigma(\mathbf{x}), j = 0, 1, \ldots, m\right\} \subseteq P_S(\mathbf{x})$$

- Truncated quadratic module:

$$\mathcal{Q}(\mathbf{g})_{2r} := \left\{\sigma_0 + \sum_{j=1}^{m} \sigma_j g_j \mid \sigma_j \in \Sigma(\mathbf{x}), \deg(\sigma_j g_j) \le 2r, j = 0, 1, \ldots, m\right\}$$

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Dual SOS relaxation

- Dual SOS relaxation:

$$\theta_r^* := \begin{cases} \sup & \lambda \\ \text{s.t.} & f - \lambda \in \mathcal{Q}(\mathbf{g})_{2r}. \end{cases}$$

$$\Updownarrow$$

$$\theta_r^* := \begin{cases} \sup & \lambda \\ \text{s.t.} & f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \\ & \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma(\mathbf{x}), \\ & \deg(\sigma_0) \le 2r, \deg(\sigma_j g_j) \le 2r, j = 1, \ldots, m. \end{cases}$$

The moment-SOS hierarchy

$$
\begin{array}{ccccc}
 & & f^* & & \\
 & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\geq$} & \\
 & \vdots & & \vdots & \\
 & \leq & & \leq & \\
\text{(moment relaxation)} & \theta_r & \text{“}=\text{”} & \theta_r^* & \text{(SOS relaxation)} \\
 & \leq & & \leq & \\
 & \vdots & & \vdots & \\
 & \leq & & \leq & \\
 & \theta_{\underline{r}} & \text{“}=\text{”} & \theta_{\underline{r}}^* &
\end{array}
$$

Asymptotical convergence and finite convergence

- Archimedean's condition: there exists $N > 0$ s.t. $N - ||\mathbf{x}||^2 \in \mathcal{Q}(\mathbf{g})$

➤ $\theta_r \uparrow f^*$ and $\theta_r^* \uparrow f^*$ when $r \to \infty$ (Lasserre, 2001);

➤ Finite convergence happens generically (Nie, 2014);

➤ We can verify global optimality by the so-called rank condition (flat extension/truncation);

➤ We can easily extract minimizers when the rank condition is satisfied.

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Scalability issue

- The size of SDP (considering the SOS problem) at relaxation order r:
  1. maximal size of PSD matrices: $\binom{n+r}{r}$
  2. number of equality constraints: $\binom{n+2r}{2r}$
- $r = 2$, $n \leq 30$ (Mosek)
- Exploiting structure:
  - quotient ring
  - symmetry
  - sparsity

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Correlative sparsity (Waki et al., 2006)

- Correlative sparsity pattern (csp) graph $G^{\text{csp}}(V, E)$:
  - $V := \{x_1, \ldots, x_n\}$
  - $\{x_i, x_j\} \in E \Longleftrightarrow x_i, x_j$ appear in the same term of f or appear in the same constraint polynomial g_k
- For each maximal clique of the csp graph $G^{\text{csp}}(V, E)$:

$$I_k \longmapsto M_r(\mathbf{y}, I_k), M_{r-d_j}(g_j\mathbf{y}, I_k)$$

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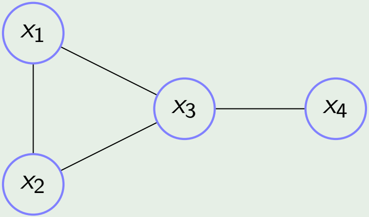
Correlative sparsity

Example

$f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$, $g_1 = 1 - x_1^2 - x_2^2 - x_3^2$, $g_2 = 1 - x_3x_4$



There are two maximal cliques: $\{x_1, x_2, x_3\}$ and $\{x_3, x_4\}$

The moment-SOS hierarchy based on correlative sparsity

- If the csp graph is chordal (otherwise we need a chordal extension), then the moment-SOS hierarchy based on correlative sparsity shares the same convergence as the standard one;
- We can still verify global optimality by the (adapted) rank condition;
- We can still extract global minimizers if certain rank conditions are satisfied;
- Significantly improve scalability if the sizes of maximal cliques of the csp graph are small (e.g. ≤ 10).

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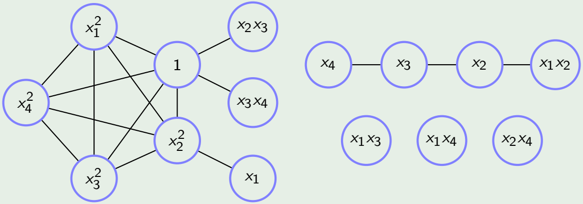
Term sparsity (Wang, Magron, and Lasserre, 2021)

- Term sparsity pattern (tsp) graph $G^{\text{tsp}}(V, E)$:
  - $V := v_r = \{1, x_1, \ldots, x_n, x_1^r, \ldots, x_n^r\}$
  - $\{\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\} \in E \iff \mathbf{x}^{\boldsymbol{\alpha}} \cdot \mathbf{x}^{\boldsymbol{\beta}} = \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in \mathrm{supp}(f) \cup \bigcup_{j=1}^{m} \mathrm{supp}(g_j) \cup v_r^2$

Term sparsity

Example

$f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$, $g_1 = 1 - x_1^2 - x_2^2 - x_3^2$, $g_2 = 1 - x_3x_4$



Term sparsity

- Assume $(G^{\rm tsp})'$ is a chordal extension of $G^{\rm tsp}$ with maximal cliques: $C_1, \ldots, C_t$

$$C_i \longmapsto M_{C_i}(\mathbf{y}), \quad i = 1, \ldots, t$$

- Decompose the moment matrix:

$$M_r(\mathbf{y}) \succeq 0 \longrightarrow M_{C_i}(\mathbf{y}) \succeq 0, \quad i = 1, \ldots, t$$

- Decompose the localizing matrix $M_{r-d_j}(\mathbf{y})$, $j = 1, \ldots, m$

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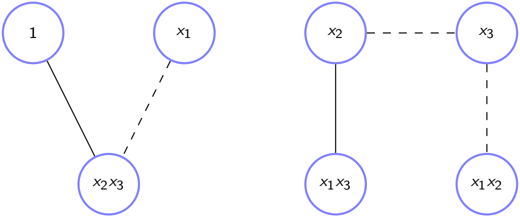
Extending to an iterative procedure

- support extension: $\mathbf{x}^{\boldsymbol{\beta}'}\mathbf{x}^{\boldsymbol{\gamma}'} = \mathbf{x}^{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\gamma}}$, $\{\mathbf{x}^{\boldsymbol{\beta}}, \mathbf{x}^{\boldsymbol{\gamma}}\} \in E \Rightarrow \{\mathbf{x}^{\boldsymbol{\beta}'}, \mathbf{x}^{\boldsymbol{\gamma}'}\} \in E$



- Iteratively perform support extension and chordal extension:

$$G^{(1)} := (G^{\text{tsp}})' \subseteq \cdots \subseteq G^{(s)} \subseteq G^{(s+1)} \subseteq \cdots$$

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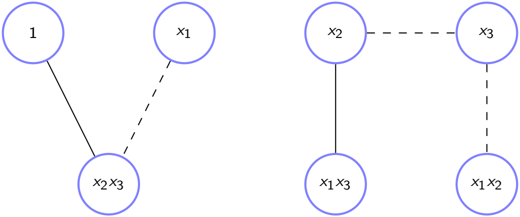

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The moment-SOS hierarchy based on term sparsity

- The maximal cliques of $G_j^{(s)}$: $C_{j,1}^{(s)}, \ldots, C_{j,t_{j,s}}^{(s)}$
- TSSOS hierarchy:

$$\theta_r^{(s)} := \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{C_{0,i}^{(s)}}(\mathbf{y}) \succeq 0, \quad i = 1, \ldots, t_{0,s}, \\ & M_{C_{j,i}^{(s)}}(g_j\mathbf{y}) \succeq 0, \quad i = 1, \ldots, t_{j,s}, j = 1, \ldots, m, \\ & y_{\mathbf{0}} = 1. \end{cases}$$

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A two-level hierarchy of lower bounds

$$
\begin{array}{ccccccc}
\theta_{\underline{r}}^{(1)} & \leq & \theta_{\underline{r}}^{(2)} & \leq & \cdots & \leq & \theta_{\underline{r}} \\
\rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} & & & & \rotatebox{-90}{$\leq$} \\
\theta_{\underline{r}+1}^{(1)} & \leq & \theta_{\underline{r}+1}^{(2)} & \leq & \cdots & \leq & \theta_{\underline{r}+1} \\
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\vdots & & \vdots & & \vdots & & \vdots \\
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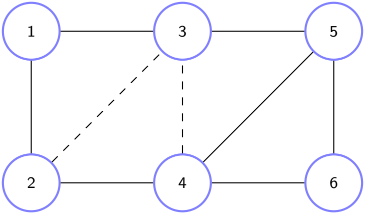
\end{array}
$$

Different choices of chordal extensions

- chordal extension:
  - maximal chordal extension
  - (approximately) smallest chordal extension



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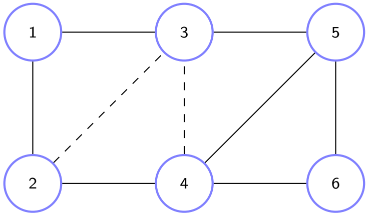

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The connection to sign symmetries

- Sign symmetries: $f = x^4y^2 + 2x^2y^4 + xy + 1$, $f(x, y) = f(-x, -y)$
- The sign symmetries induce a partition of monomial bases:

$\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}$ belong to the same block $\iff$ $\mathbf{x}^{\alpha+\beta}$ is invariant under the sign symmetries

Theorem (Wang, Magron, and Lasserre, 2021)

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Combining correlative sparsity with term sparsity

- The CS-TSSOS hierarchy:

1. Decomposing the variables with respect to the maximal cliques of the csp graph;
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SONC decomposition and second order cone representation

The second order cone representation of SONC cones

- Second order cone: $\mathbb{S}_+^2 := \{(a, b, c) \in \mathbb{R}^3 \mid \begin{bmatrix} a & b \\ b & c \end{bmatrix} \succeq 0\}$
- SONC cone: Given $\mathscr{A}, \mathscr{B}_1 \subseteq (2\mathbb{N})^n$ and $\mathscr{B}_2 \subseteq \mathbb{N}^n \backslash (2\mathbb{N})^n$,

$$\mathrm{SONC}_{\mathscr{A},\mathscr{B}_1,\mathscr{B}_2} := \{(\mathbf{c}_{\mathscr{A}}, \mathbf{d}_{\mathscr{B}_1}, \mathbf{d}_{\mathscr{B}_2}) \in \mathbb{R}_+^{|\mathscr{A}|} \times \mathbb{R}_+^{|\mathscr{B}_1|} \times \mathbb{R}^{|\mathscr{B}_2|} \mid \sum_{\alpha \in \mathscr{A}} c_\alpha \mathbf{x}^\alpha - \sum_{\beta \in \mathscr{B}_1 \cup \mathscr{B}_2} d_\beta \mathbf{x}^\beta \in \mathrm{SONC}\}$$

Theorem (Wang and Magron, 2020)

Any SONC cone admits a second order cone representation.

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Circuit polynomial and geometric mean inequality

- $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ ($d_{\boldsymbol{\beta}} > 0$) a circuit polynomial
- There exist unique barycentric coordinates $(\lambda_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathscr{A}} \subseteq \mathbb{R}^{+}$ s.t.

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}} = 1, \text{ and } \boldsymbol{\beta} = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}\boldsymbol{\alpha}$$

- Then,

$$f \geq 0 \iff \prod_{\boldsymbol{\alpha}\in\mathscr{A}} \left(\frac{c_{\boldsymbol{\alpha}}}{\lambda_{\boldsymbol{\alpha}}}\right)^{\lambda_{\boldsymbol{\alpha}}} \geq d_{\boldsymbol{\beta}} \iff \prod_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} \geq \prod_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} d_{\boldsymbol{\beta}}$$

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$$f \geq 0 \iff \prod_{\boldsymbol{\alpha}\in\mathscr{A}} \left(\frac{c_{\boldsymbol{\alpha}}}{\lambda_{\boldsymbol{\alpha}}}\right)^{\lambda_{\boldsymbol{\alpha}}} \geq d_{\boldsymbol{\beta}} \iff \prod_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} \geq \prod_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} d_{\boldsymbol{\beta}}$$

Mediated sets and sum of binomial squares

- $\mathscr{A}$-mediated set: $M \subseteq \mathbb{Q}^n$, for each $\mathbf{w} \in M$,

$$\exists \mathbf{u} \neq \boldsymbol{v} \in M \cup \mathscr{A}, \text{ s.t. } \mathbf{w} = \frac{1}{2}(\mathbf{u} + \boldsymbol{v})$$

Theorem (Wang and Magron, 2020)

A circuit polynomial $f = \sum_{\alpha \in \mathscr{A}} c_\alpha \mathbf{x}^\alpha - d_\beta \mathbf{x}^\beta \in \mathbb{R}[\mathbf{x}]$ ($d_\beta > 0$) is nonnegative if and only if $f = \sum_{\mathbf{w}_i \in M} (a_i \mathbf{x}^{\frac{1}{2}\mathbf{u}_i} - b_i \mathbf{x}^{\frac{1}{2}\boldsymbol{v}_i})^2$, $a_i, b_i \in \mathbb{R}$ for any $\mathscr{A}$-mediated set M containing β.

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Correspondences

construct a second order cone representation for a SONC cone $\mathcal{S}$

$$\Updownarrow$$

rewrite any circuit polynomial $f \in \mathcal{S}$ as a sum of binomial squares

$$\Updownarrow$$

for any lattice point $\boldsymbol{\beta}$, find an $\mathscr{A}$-mediated set M containing $\boldsymbol{\beta}$

$$\Updownarrow$$

rewrite $\prod_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} \geq \prod_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}} d_{\boldsymbol{\beta}}$ as a set of quadratic inequalities

Optimal second order cone representations for geometric mean inequalities

Question

Given an inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\sum_{i=1}^m s_i}$ $(x_i \in \mathbb{R}^+, s_i \in \mathbb{N}^*)$, construct an equivalent representation using as few quadratic inequalities as possible.

Example: $x_1^3 x_2^8 \geq x_3^{11} \iff x_2x_4 \geq x_3^2, x_5x_6 \geq x_4^2, x_1x_3 \geq x_5^2, x_3x_5 \geq x_6^2$

- $L(s_1, \ldots, s_m)$: minimum number of quadratic inequalities
  - $L(s_1, s_2) = \lceil \log_2 (s_1 + s_2) \rceil$
  - $L(s_1, \ldots, s_m) \geq \lceil \log_2 (\sum_{i=1}^m s_i) \rceil$

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A heuristic algorithm for (nearly) optimal second order cone representations

$$x_1^{s_1}\cdots x_m^{s_m}\ge x_{m+1}^{\sum_{i=1}^m s_i}$$

$$\Updownarrow$$

$$x_1^{s_1}\cdots x_i^{s_i-s_i}\cdots x_j^{s_j-s_i}\cdots x_m^{s_m}x_{m+2}^{2s_i}\ge x_{m+1}^{\sum_{i=1}^m s_i},\ x_ix_j\ge x_{m+2}^2$$

$$\Updownarrow$$

$$\vdots$$

$$x_{i_1}x_{j_1}\ge x_{m+1}^2,\dots,x_{i_n}x_{j_n}\ge x_{m+n}^2,\ i_k,j_k\in\{1,2,\dots,m+n\}$$

Software and numerical experiments

- TSSOS:

https://github.com/wangjie212/TSSOS

- SONCSOCP:

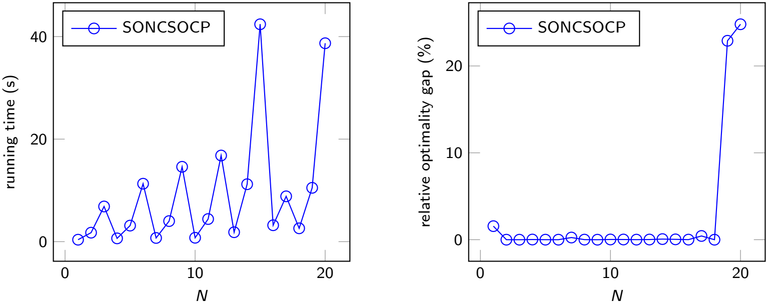
https://github.com/wangjie212/SONCSOCP

Optimal power flow (AC-OPF)

n	m+m'	CS ($r = 2$)				CS+TS ($r = 2, s = 1$)			
		mb	opt	time (s)	gap	mb	opt	time (s)	gap
12	28	28	1.1242e4	0.21	0.00%	22	1.1242e4	0.09	0.00%
20	55	28	1.7543e4	0.56	0.05%	22	1.7543e4	0.30	0.05%
72	297	45	4.9927e3	4.43	0.07%	22	4.9920e3	2.69	0.08%
114	315	120	7.6943e4	94.9	0.00%	39	7.6942e4	14.8	0.00%
344	1325	253	—	—	—	73	1.0470e5	169	0.50%
348	1809	253	—	—	—	34	1.2096e5	201	0.03%
766	3322	153	3.3072e6	585	0.68%	44	3.3042e6	33.9	0.77%
1112	4613	496	—	—	—	31	7.2396e4	410	0.25%
4356	18257	378	—	—	—	27	1.3953e6	934	0.51%
6698	29283	1326	—	—	—	76	5.9858e5	1886	0.47%

Optimization with polynomials of high degree

number of variables: 10, degree: $20 \sim 50$, number of terms: $30 \sim 300$



Summary

- Term sparsity opens a new window for exploiting sparsity in polynomial optimization
- When appropriate sparsity patterns are accessible, significantly improve the scalability of the moment-SOS hierarchy
- SONC decompositions work for sparse polynomials of high degree
- Further investigations on $L(s_1, \ldots, s_m)$

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Main references

- **Jie Wang**, *Nonnegative Polynomials and Circuit Polynomials*, SIAM Journal on Applied Algebra and Geometry, 2022.
- **Jie Wang**, Victor Magron and Jean B. Lasserre, *TSSOS: A Moment-SOS hierarchy that exploits term sparsity*, SIAM Journal on Optimization, 2021.
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- **Jie Wang**, Victor Magron, Jean B. Lasserre and Ngoc H. A. Mai, *CS-TSSOS: Correlative and term sparsity for large-scale polynomial optimization*, arXiv:2005.02828, 2020.

Thanks for your attention!

https://wangjie212.github.io/jiewang