Nonnegative polynomials and circuit polynomials

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Certify nonnegativity of polynomials

Nonnegative Polynomials

Problem

Given a multivariate polynomial f, decide if f is nonnegative and certify its nonnegativity if it is.

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Given a multivariate polynomial f, decide if f is nonnegative and certify its nonnegativity if it is.

- Certifying nonnegativity of multivariate polynomials is a central problem in real algebraic geometry which has applications in polynomial optimization and many other fields such as control, engineering, combinatorics, and physics.
- Generally, deciding nonnegativity of multivariate polynomials is an NP-hard problem.

Sums of squares

A classical approach for certifying nonnegativity of polynomials is the use of sums of squares.

Sums of squares

Given a polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$, if there exist polynomials $f_1, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ such that

$$f = \sum_{i=1}^{m} f_i^2,$$

then we say f is a sum of squares (SOS).

SOS decompositions

- Though the existence of SOS decompositions is only a sufficient condition of nonnegativity, it gives a computational approach to certify nonnegativity.

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- M: a monomial basis
  f admits an SOS decomposition
  $\iff \exists$ a positive semidefinite matrix Q s.t. $f = M^T Q M$
  $\rightsquigarrow$ effectively solved by semidefinite programming (SDP)
  (Parrilo 2000, Lasserre 2001)

Exploiting Sparsity

- f: n variables, $2d$ degree, SDP: $\binom{n+d}{n}$
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- The size of the corresponding semidefinite program problem grows rapidly as the size of the polynomial increases.
- To deal with large polynomials, sparsity must be exploited.
- Newton polytopes (Reznick, 1978), correlative sparsity patterns (Waki et al., 2006), sign-symmetries (Löfberg, 2009), the facial reduction (Permenter and Parrilo, 2014), the split property (Dai and Xia, 2015), minimal coordinate projections (Permenter and Parrilo, 2015), cross sparsity patterns (Wang, Li and Xia, 2019).

Certify nonnegativity via SONC decompositions

Sparse nonnegative polynomial

An n-variate dense polynomial of degree d consists of $\binom{n+d}{n}$ monomials.
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A sparse polynomial has $\ll \binom{n+d}{n}$ terms.

Problem

Does there exist a method for certifying nonnegativity of sparse polynomials, whose complexity depends on the number of terms, not on degrees?

Circuit polynomials

Trellis: $\mathscr{A} \subseteq (2\mathbb{N})^n$ comprises the vertices of a simplex

Definition (Iliman and Wolff, 2016)

Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a circuit polynomial if it is of the form

$$f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}},$$

and satisfies:

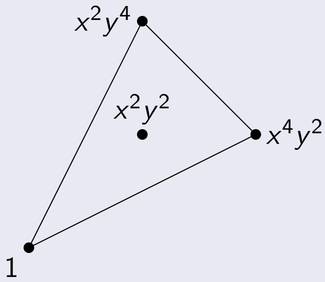
(1) $c_{\boldsymbol{\alpha}} > 0$ for $\boldsymbol{\alpha} \in \mathscr{A}$;

(2) $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^{\circ}$.

Circuit polynomials

Example (Motzkin's polynomial)

The Motzkin's polynomial $M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is a nonnegative circuit polynomial.



Sums of nonnegative circuit polynomials

A polynomial decomposes into a sum of squares of polynomials (SOS) $\Longrightarrow$ it is nonnegative

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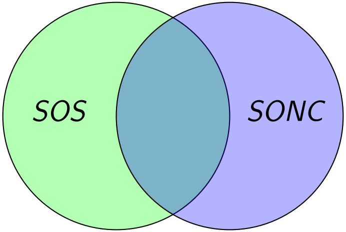


Sums of nonnegative circuit polynomials

Question 1 : Under which conditions, a nonnegative polynomial admits an SONC decomposition? (A Hilbert-style theorem for the SONC case)

Sums of nonnegative circuit polynomials

Question 1 : Under which conditions, a nonnegative polynomial admits an SONC decomposition? (A Hilbert-style theorem for the SONC case)

Question 2 : How to efficiently compute an SONC decomposition of a nonnegative polynomial if there exists?

The case of one negative term

Theorem (Wang, 2018)

Let $f = \sum_{\alpha \in \mathscr{A}} c_\alpha \mathbf{x}^\alpha - d\mathbf{x}^\beta \in \mathbb{R}[\mathbf{x}]$ with $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\beta \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n$ and $c_\alpha > 0$ for $\alpha \in \mathscr{A}$. Then f is nonnegative if and only if f admits an SONC decomposition.

The case of multiple negative terms

Theorem (Wang, 2018)

Let $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ with $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B} \subseteq \mathrm{New}(f)^{\circ} \cap \mathbb{N}^n$ and $c_{\boldsymbol{\alpha}} > 0$ for $\boldsymbol{\alpha} \in \mathscr{A}$. Assume that

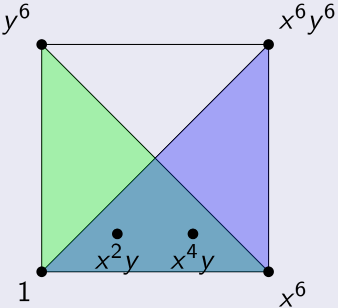
- $\mathrm{New}(f)$ is simple at some vertex,
- all $\boldsymbol{\beta}$'s lie in the same side of every hyperplane determined by points among $\mathscr{A}$,
- there exists a point $\mathbf{v} = (v_k) \in (\mathbb{R}^*)^n$ such that $d_{\boldsymbol{\beta}} \mathbf{v}^{\boldsymbol{\beta}} > 0$ for all $\boldsymbol{\beta}$.

Then f is nonnegative if and only if f admits an SONC decomposition.

An example

Example

Let $f = 1 + x^6 + y^6 + x^6y^6 - x^2y - 2x^4y$ which is nonnegative. Then $f \in \mathrm{SONC}$.



Compute SONC decompositions via second-order cone programming

The connection with SOS

- For a subset $M \subseteq \mathbb{N}^n$, $\overline{A}(M) := \{\frac{1}{2}(\mathbf{u}+\mathbf{v}) \mid \mathbf{u} \neq \mathbf{v}, \mathbf{u}, \mathbf{v} \in M \cap (2\mathbb{N})^n\}$.
- For a trellis $\mathscr{A}$, M is an $\mathscr{A}$-mediated set if $\mathscr{A} \subseteq M \subseteq \overline{A}(M) \cup \mathscr{A}$.

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Theorem (Reznick, 1989; Iliman and Wolff, 2016)

Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$ be a nonnegative circuit polynomial with $\mathscr{A}$ a trellis. Then f is a sum of squares if and only if there exists an $\mathscr{A}$-mediated set containing $\boldsymbol{\beta}$. Moreover, suppose that $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set $M = \{\mathbf{u}_i\}_{i=1}^{s}$. For each $\mathbf{u}_i \in M \backslash \mathscr{A}$, let $\mathbf{u}_i = \frac{1}{2}(\mathbf{u}_{p(i)} + \mathbf{u}_{q(i)})$. Then f is a sum of binomial squares and $f = \sum_{\mathbf{u}_i \in M\backslash\mathscr{A}} (a_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{p(i)}} - b_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{q(i)}})^2$, $a_i, b_i \in \mathbb{R}$.

$\mathscr{A}$-rational mediated sets

For a subset $M \subseteq \mathbb{Q}^n$, $\widetilde{A}(M) := \{\frac{1}{2}(\mathbf{u}+\mathbf{v}) \mid \mathbf{u} \neq \mathbf{v}, \mathbf{u}, \mathbf{v} \in M\}$. Let $\mathscr{A} \subseteq \mathbb{Q}^n$ comprise the vertices of a simplex. We say that M is an $\mathscr{A}$-rational mediated set if $\mathscr{A} \subseteq M \subseteq \widetilde{A}(M) \cup \mathscr{A}$.

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Theorem (Wang, 2019)

For a trellis $\mathscr{A} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ and a lattice point $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^\circ$, there always exists an $\mathscr{A}$-rational mediated set $M_{\mathscr{A}\boldsymbol{\beta}}$ containing $\boldsymbol{\beta}$ such that the denominators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}$ are odd numbers and the numerators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}} \backslash \{\boldsymbol{\beta}\}$ are even numbers.

Remark: Actually we have an algorithm to compute an $\mathscr{A}$-rational mediated set $M_{\mathscr{A}\boldsymbol{\beta}}$ containing $\boldsymbol{\beta}$ with the desired property.

$\mathscr{A}$-rational mediated sets

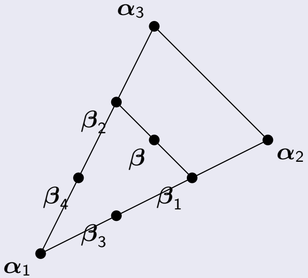
Theorem (Wang, 2019)

Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$ be a circuit polynomial and assume that $M_{\mathscr{A}\boldsymbol{\beta}} = \{\mathbf{u}_i\}_{i=1}^{s}$ is an $\mathscr{A}$-rational mediated set containing $\boldsymbol{\beta}$ such that the denominators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}$ are odd numbers and the numerators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}\backslash\{\boldsymbol{\beta}\}$ are even numbers. For each $\mathbf{u}_i \in M_{\mathscr{A}\boldsymbol{\beta}}\backslash\mathscr{A}$, let $\mathbf{u}_i = \frac{1}{2}(\mathbf{u}_{p(i)} + \mathbf{u}_{q(i)})$. Then f is nonnegative if and only if f can be written as
$f = \sum_{\mathbf{u}_i\in M_{\mathscr{A}\boldsymbol{\beta}}\backslash\mathscr{A}} (a_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{p(i)}} - b_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{q(i)}})^2$, $a_i, b_i \in \mathbb{R}$.

An example

Example

Let $f = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ be the Motzkin's polynomial and $\mathscr{A} = \{\boldsymbol{\alpha}_1 = (0,0), \boldsymbol{\alpha}_2 = (4,2), \boldsymbol{\alpha}_3 = (2,4)\}$, $\boldsymbol{\beta} = (2,2)$. Then $M = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$ is an $\mathscr{A}$-rational mediated set containing $\boldsymbol{\beta}$.



By a simple computation, we have

$$f = \frac{3}{2}(x^{\frac{2}{3}}y^{\frac{4}{3}} - x^{\frac{4}{3}}y^{\frac{2}{3}})^2 + (xy^2 - x^{\frac{1}{3}}y^{\frac{2}{3}})^2 + \frac{1}{2}(x^{\frac{2}{3}}y^{\frac{4}{3}} - 1)^2 + (x^2y - x^{\frac{2}{3}}y^{\frac{1}{3}})^2 + \frac{1}{2}(x^{\frac{4}{3}}y^{\frac{2}{3}} - 1)^2.$$

Compute SONC decompositions via second-order cone programming

A polynomial $f \in \mathrm{SONC} \iff f$ is a sum of "binomial" squares with rational exponents

$\rightsquigarrow$ efficiently solved by a 2×2-blocked SDP problem (second-order cone programming (SOCP))

Second-order cone representations of SONC cones

Second-order cones

The n-dimensional standard second-order cone is defined as

$$\mathcal{C} := \{(\mathbf{x}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : ||\mathbf{x}||_2 \le t\},$$

and an n-dimensional second-order cone is

$$\mathcal{Q} := \{\mathbf{x} \in \mathbb{R}^m : ||A\mathbf{x} + \mathbf{b}||_2 \le \mathbf{c}^T\mathbf{x} + d\},$$

where $A \in \mathbb{R}^{(n-1)\times m}, \mathbf{b} \in \mathbb{R}^{n-1}, \mathbf{c} \in \mathbb{R}^m, d \in \mathbb{R}$.

Remark: The optimization problem over second-order cones can be solved more efficiently than semidefinite programming.

Second-order cones

Example

$$\mathbb{S}_{+}^{2} := \{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2} \mid \begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ is positive semidefinite}\}$$

is a 3-dimensional second-order cone.

Second-order cone lifts of convex cones

$\mathcal{Q}^k = \mathcal{Q} \times \cdots \mathcal{Q}$: the Cartesian product of k copies of a second-order cone $\mathcal{Q}$

Definition

A convex cone $C \subseteq \mathbb{R}^m$ has a second-order cone lift of size k (or simply a $\mathcal{Q}^k$-lift) if it can be written as the projection of a slice of $\mathcal{Q}^k$, that is, there is a subspace L of $\mathcal{Q}^k$ and a linear map $\pi\colon \mathcal{Q}^k \to \mathbb{R}^m$ such that $C = \pi(\mathcal{Q}^k \cap L)$.

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Theorem (Fawzi, 2018)

The cone $\mathrm{SOS}_{n,2d}$ does not admit any second-order cone lift except in the case $(n, 2d) = (1, 2)$.

$(\mathbb{S}^2_+)^k$-lifts of SONC cones

Given $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B}_1 \subseteq \mathrm{conv}(\mathscr{A}) \cap (2\mathbb{N})^n$ and $\mathscr{B}_2 \subseteq \mathrm{conv}(\mathscr{A}) \cap (\mathbb{N}^n \backslash (2\mathbb{N})^n)$ such that $\mathscr{A} \cap \mathscr{B}_1 = \varnothing$, define

$$\begin{aligned}\mathrm{SONC}_{\mathscr{A},\mathscr{B}_1,\mathscr{B}_2} :=& \{(\mathbf{c}_{\mathscr{A}}, \mathbf{d}_{\mathscr{B}_1}, \mathbf{d}_{\mathscr{B}_2}) \in \mathbb{R}_+^{|\mathscr{A}|} \times \mathbb{R}_+^{|\mathscr{B}_1|} \times \mathbb{R}^{|\mathscr{B}_2|} \\ & \mid \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \mathscr{B}_1 \cup \mathscr{B}_2} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{SONC}\},\end{aligned}$$

which is a convex cone.

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Theorem (Wang, 2019)

For $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B}_1 \subseteq \mathrm{conv}(\mathscr{A}) \cap (2\mathbb{N})^n$ and $\mathscr{B}_2 \subseteq \mathrm{conv}(\mathscr{A}) \cap (\mathbb{N}^n \backslash (2\mathbb{N})^n)$ such that $\mathscr{A} \cap \mathscr{B}_1 = \varnothing$, the convex cone $\mathrm{SONC}_{\mathscr{A},\mathscr{B}_1,\mathscr{B}_2}$ admits an $(\mathbb{S}^2_+)^k$-lift for some $k \in \mathbb{N}$.

Comparison between SONC and SOS

	SONC	SOS
sparsity	maintain	not maintain
computation	REP/SOCP	SDP
complexity	depends on n,m	depends on n,d
second-order cone representation	exist	not exist
...	...	...

Table: n: number of variables, d: degree, m: number of terms

Summary

- SONC decompositions provide a new computational framework for certificates of sparse nonnegativity and sparse polynomial optimization.
- A partial Hilbert-style theorem for the SONC case is given.
- We can compute SONC decompositions via SOCP.
- The SONC cone admits a second-order cone representation.

Thank you!