A Second Order Cone Characterization for Sums of Nonnegative Circuits

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Content

Background on SONC polynomials

Nonnegative Polynomials

Problem

Given a multivariate polynomial f, decide whether f is nonnegative and certify its nonnegativity if it is.

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- Certifying nonnegativity of multivariate polynomials is a central problem in real algebraic geometry and has applications in polynomial optimization, control, combinatorics and so on.

Nonnegative Polynomials

Problem

Given a multivariate polynomial f, decide whether f is nonnegative and certify its nonnegativity if it is.

- Certifying nonnegativity of multivariate polynomials is a central problem in real algebraic geometry and has applications in polynomial optimization, control, combinatorics and so on.
- Generally, deciding nonnegativity of multivariate polynomials is NP-hard.

Sums of squares

A classical approach for certifying nonnegativity of polynomials is using sums of squares.

Sums of squares

Given a polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$, if there exist polynomials $f_1, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ such that

$$f = \sum_{i=1}^{m} f_i^2,$$

then we say f is a sum of squares (SOS).

Note: The computation of SOS decompositions for a given polynomial can be done via semidefinite programming (SDP).

Questions

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- **Question 2**: If the answer is yes, how can we efficiently compute such a nonnegativity certificate for a given polynomial?

Circuit polynomials

Trellis: $\mathscr{A} \subseteq (2\mathbb{N})^n$ comprises the vertices of a simplex

Definition (Iliman and Wolff, 2016)

Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a circuit polynomial if it is of the form

$$f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}},$$

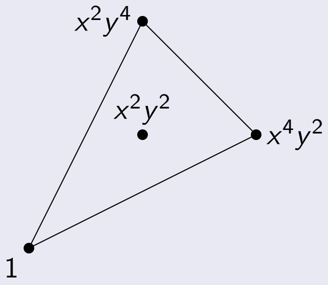
and satisfies:

1. $c_{\boldsymbol{\alpha}} > 0$ for $\boldsymbol{\alpha} \in \mathscr{A}$;
2. $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^{\circ}$.

Circuit polynomials

Example (Motzkin's polynomial)

The Motzkin's polynomial $M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is a nonnegative circuit polynomial.



SONC polynomials

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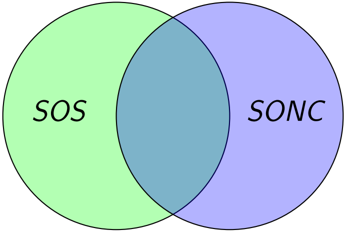

SONC decompositions preserve sparsity of polynomials

For $f \in \mathbb{R}[\mathbf{x}]$, let

$$\Lambda(f) := \{\boldsymbol{\alpha} \in \mathrm{supp}(f) \mid \boldsymbol{\alpha} \in (2\mathbb{N})^n \text{ and } c_{\boldsymbol{\alpha}} > 0\}$$

and

$$\Gamma(f) := \mathrm{supp}(f) \backslash \Lambda(f)$$

such that we can write $f = \sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \Gamma(f)} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$.

For each $\boldsymbol{\beta} \in \Gamma(f)$, let

$$\mathscr{F}(\boldsymbol{\beta}) := \{\Delta \mid \Delta \text{ is a simplex, } \boldsymbol{\beta} \in \Delta^{\circ}, V(\Delta) \subseteq \Lambda(f)\}.$$

SONC decompositions preserve sparsity of polynomials

Theorem (Wang, 2018)

Let $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. If $f \in \mathrm{SONC}$, then f admits a SONC decomposition:

$$f = \sum_{\boldsymbol{\beta}\in\Gamma(f)} \sum_{\Delta\in\mathscr{F}(\boldsymbol{\beta})} f_{\boldsymbol{\beta}\Delta} + \sum_{\boldsymbol{\alpha}\in\tilde{\mathscr{A}}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}},$$

where $f_{\boldsymbol{\beta}\Delta}$ is a nonnegative circuit polynomial supported on $V(\Delta)\cup\{\boldsymbol{\beta}\}$ for each Δ and $\tilde{\mathscr{A}} = \{\boldsymbol{\alpha}\in\Lambda(f) \mid \boldsymbol{\alpha}\notin \cup_{\boldsymbol{\beta}\in\Gamma(f)}\cup_{\Delta\in\mathscr{F}(\boldsymbol{\beta})} V(\Delta)\}$.

Remark: This is dramatically different from the SOS case in which extra monomials are needed in general.

SONC polynomials and sums of binomial squares

Circuit polynomials and sums of binomial squares

- For a subset $M \subseteq \mathbb{N}^n$, let $\overline{A}(M) := \{\frac{1}{2}(\mathbf{u} + \boldsymbol{v}) \mid \mathbf{u} \neq \boldsymbol{v}, \mathbf{u}, \boldsymbol{v} \in M \cap (2\mathbb{N})^n\}$.
- For a trellis $\mathscr{A}$, M is an $\mathscr{A}$-mediated set if $\mathscr{A} \subseteq M \subseteq \overline{A}(M) \cup \mathscr{A}$.

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Theorem (Reznick, 1989; Iliman and Wolff, 2016)

Let $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$ be a nonnegative circuit polynomial with $\mathscr{A}$ a trellis. Then f is a sum of binomial squares if and only if there exists an $\mathscr{A}$-mediated set containing $\boldsymbol{\beta}$. More specifically, suppose that $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set $M = \{\mathbf{u}_i\}_{i=1}^{s}$. For each $\mathbf{u}_i \in M \backslash \mathscr{A}$, let $\mathbf{u}_i = \frac{1}{2}(\mathbf{u}_{p(i)} + \mathbf{u}_{q(i)})$. Then f is a sum of binomial squares and $f = \sum_{\mathbf{u}_i \in M \backslash \mathscr{A}} (a_i \mathbf{x}^{\frac{1}{2}\mathbf{u}_{p(i)}} - b_i \mathbf{x}^{\frac{1}{2}\mathbf{u}_{q(i)}})^2$, $a_i, b_i \in \mathbb{R}$.

Theorem (Reznick, 1989; Iliman and Wolff, 2016)

Theorem (Reznick, 1989; Iliman and Wolff, 2016) inspires us to leverage sums of binomial squares to compute SONC decompositions. However, there are two obstacles:

- There may not exist such an $\mathscr{A}$-mediated set containing a given lattice point;
- Even if such a set exists, there is no efficient algorithm to compute it.

$\mathscr{A}$-rational mediated sets

For $M \subseteq \mathbb{Q}^n$, let $\widetilde{A}(M) := \{\frac{1}{2}(\mathbf{u} + \boldsymbol{v}) \mid \mathbf{u} \neq \boldsymbol{v}, \mathbf{u}, \boldsymbol{v} \in M\}$.
Let $\mathscr{A}$ be a trellis. We say that M is an $\mathscr{A}$-rational mediated set if $\mathscr{A} \subseteq M \subseteq \widetilde{A}(M) \cup \mathscr{A}$.

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Let $\mathscr{A}$ be a trellis. We say that M is an $\mathscr{A}$-rational mediated set if $\mathscr{A} \subseteq M \subseteq \widetilde{A}(M) \cup \mathscr{A}$.

Theorem (Wang and Magron, 2020)

Given a trellis $\mathscr{A}$ and a lattice point $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^{\circ}$, there is an algorithm to compute an $\mathscr{A}$-rational mediated set $M_{\mathscr{A}\boldsymbol{\beta}}$ containing $\boldsymbol{\beta}$ such that the denominators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}$ are odd numbers and the numerators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}} \backslash \{\boldsymbol{\beta}\}$ are even numbers.

Circuit polynomials and sums of binomial squares

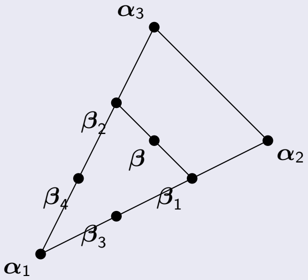
Theorem (Wang and Magron, 2020)

Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$ be a circuit polynomial and assume that $M_{\mathscr{A}\boldsymbol{\beta}} = \{\mathbf{u}_i\}_{i=1}^{s}$ is an $\mathscr{A}$-rational mediated set containing $\boldsymbol{\beta}$ such that the denominators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}$ are odd numbers and the numerators of coordinates of points in $M_{\mathscr{A}\boldsymbol{\beta}}\backslash\{\boldsymbol{\beta}\}$ are even numbers. For each $\mathbf{u}_i \in M_{\mathscr{A}\boldsymbol{\beta}}\backslash\mathscr{A}$, let $\mathbf{u}_i = \frac{1}{2}(\mathbf{u}_{p(i)} + \mathbf{u}_{q(i)})$. Then f is nonnegative if and only if f can be written as
$f = \sum_{\mathbf{u}_i\in M_{\mathscr{A}\boldsymbol{\beta}}\backslash\mathscr{A}}(a_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{p(i)}} - b_i\mathbf{x}^{\frac{1}{2}\mathbf{u}_{q(i)}})^2$, $a_i, b_i \in \mathbb{R}$.

An example

Example

Let $f = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ be Motzkin's polynomial and $\mathscr{A} = \{\boldsymbol{\alpha}_1 = (0,0), \boldsymbol{\alpha}_2 = (4,2), \boldsymbol{\alpha}_3 = (2,4)\}$, $\boldsymbol{\beta} = (2,2)$. Then $M = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$ is an $\mathscr{A}$-rational mediated set containing $\boldsymbol{\beta}$.



By a simple computation, we have
$f = \frac{3}{2}(x^{\frac{2}{3}}y^{\frac{4}{3}} - x^{\frac{4}{3}}y^{\frac{2}{3}})^2 + (xy^2 - x^{\frac{1}{3}}y^{\frac{2}{3}})^2 + \frac{1}{2}(x^{\frac{2}{3}}y^{\frac{4}{3}} - 1)^2 + (x^2y - x^{\frac{2}{3}}y^{\frac{1}{3}})^2 + \frac{1}{2}(x^{\frac{4}{3}}y^{\frac{2}{3}} - 1)^2.$

SONC polynomials and sums of binomial squares

Theorem (Wang and Magron, 2020)

Let $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. For every $\boldsymbol{\beta} \in \Gamma(f)$ and every $\Delta \in \mathscr{F}(\boldsymbol{\beta})$, let $M_{\boldsymbol{\beta}\Delta}$ be a $V(\Delta)$-rational mediated set containing $\boldsymbol{\beta}$ such that the denominators of coordinates of points in $M_{\boldsymbol{\beta}\Delta}$ are odd numbers and the numerators of coordinates of points in $M_{\boldsymbol{\beta}\Delta}\backslash\{\boldsymbol{\beta}\}$ are even numbers. Let $M = \cup_{\boldsymbol{\beta}\in\Gamma(f)} \cup_{\Delta\in\mathscr{F}(\boldsymbol{\beta})} M_{\boldsymbol{\beta}\Delta}$. For each $\mathbf{u} \in M\backslash\Lambda(f)$, let $\mathbf{u} = \frac{1}{2}(\boldsymbol{v}_{\mathbf{u}} + \mathbf{w}_{\mathbf{u}}), \boldsymbol{v}_{\mathbf{u}} \neq \mathbf{w}_{\mathbf{u}} \in M$. Let $\tilde{\mathscr{A}} = \{\boldsymbol{\alpha} \in \Lambda(f) \mid \boldsymbol{\alpha} \notin \cup_{\boldsymbol{\beta}\in\Gamma(f)} \cup_{\Delta\in\mathscr{F}(\boldsymbol{\beta})} V(\Delta)\}$. Then $f \in \mathrm{SONC}$ iff f can be written as $f = \sum_{\mathbf{u}\in M\backslash\Lambda(f)} (a_{\mathbf{u}}\mathbf{x}^{\frac{1}{2}\boldsymbol{v}_{\mathbf{u}}} - b_{\mathbf{u}}\mathbf{x}^{\frac{1}{2}\mathbf{w}_{\mathbf{u}}})^2 + \sum_{\boldsymbol{\alpha}\in\tilde{\mathscr{A}}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$, $a_{\mathbf{u}}, b_{\mathbf{u}} \in \mathbb{R}$.

Second order cones

An n-dimensional second order cone (SOC) is

$$\mathcal{Q} := \{\mathbf{x} \in \mathbb{R}^m : ||A\mathbf{x} + \mathbf{b}||_2 \leq \mathbf{c}^T\mathbf{x} + d\},$$

where $A \in \mathbb{R}^{(n-1)\times m}, \mathbf{b} \in \mathbb{R}^{n-1}, \mathbf{c} \in \mathbb{R}^m, d \in \mathbb{R}$.

Remark: The optimization problem over second order cones can be solved more efficiently than semidefinite programming.

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Example

$$\mathbb{S}_+^2 := \{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2} \mid \begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ is positive semidefinite}\}$$

is a 3-dimensional second order cone.

Second order cone lifts of convex cones

$\mathcal{Q}^k = \mathcal{Q} \times \cdots \mathcal{Q}$: the Cartesian product of k copies of a second order cone $\mathcal{Q}$

Definition

A convex cone $C \subseteq \mathbb{R}^m$ has a second order cone lift of size k (or simply a $\mathcal{Q}^k$-lift) if it can be written as the projection of a slice of $\mathcal{Q}^k$, that is, there is a subspace L of $\mathcal{Q}^k$ and a linear map $\pi \colon \mathcal{Q}^k \to \mathbb{R}^m$ such that $C = \pi(\mathcal{Q}^k \cap L)$.

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Theorem (Fawzi, 2018)

The cone $\mathrm{SOS}_{n,2d}$ does not admit any second order cone lift except in the case $(n, 2d) = (1, 2)$.

$(\mathbb{S}_+^2)^k$-lifts of SONC cones

Given $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B}_1 \subseteq \mathrm{conv}(\mathscr{A}) \cap (2\mathbb{N})^n$ and $\mathscr{B}_2 \subseteq \mathrm{conv}(\mathscr{A}) \cap (\mathbb{N}^n \backslash (2\mathbb{N})^n)$ such that $\mathscr{A} \cap \mathscr{B}_1 = \varnothing$, define the SONC cone supported on $\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2$ as

$$\begin{aligned}\mathrm{SONC}_{\mathscr{A},\mathscr{B}_1,\mathscr{B}_2} :=& \{(\mathbf{c}_{\mathscr{A}}, \mathbf{d}_{\mathscr{B}_1}, \mathbf{d}_{\mathscr{B}_2}) \in \mathbb{R}_+^{|\mathscr{A}|} \times \mathbb{R}_+^{|\mathscr{B}_1|} \times \mathbb{R}^{|\mathscr{B}_2|} \\ & \mid \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \mathscr{B}_1 \cup \mathscr{B}_2} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathrm{SONC}\}.\end{aligned}$$

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Theorem (Wang and Magron, 2020)

For $\mathscr{A} \subseteq (2\mathbb{N})^n$, $\mathscr{B}_1 \subseteq \mathrm{conv}(\mathscr{A}) \cap (2\mathbb{N})^n$ and $\mathscr{B}_2 \subseteq \mathrm{conv}(\mathscr{A}) \cap (\mathbb{N}^n \backslash (2\mathbb{N})^n)$ such that $\mathscr{A} \cap \mathscr{B}_1 = \varnothing$, the SONC cone $\mathrm{SONC}_{\mathscr{A},\mathscr{B}_1,\mathscr{B}_2}$ admits an $(\mathbb{S}_+^2)^k$-lift for some $k \in \mathbb{N}$.

SONC optimization via second order cone programming

SONC optimization

Consider the unconstrained polynomial optimization problem:

$$(\text{UPOP}): \quad \xi^* := \begin{cases} \mathsf{sup} & \xi \\ \text{s.t.} & f(\mathbf{x}) - \xi \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

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Replacing the nonnegativity condition by SONC to obtain:

$$(\text{SONC}): \quad \xi_{sonc} := \begin{cases} \sup & \xi \\ \text{s.t.} & f(\mathbf{x}) - \xi \in \text{SONC}. \end{cases}$$

PN-polynomials

Suppose $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. If $d_{\boldsymbol{\beta}} > 0$ for all $\boldsymbol{\beta} \in \Gamma(f)$, then we call f a PN-polynomial.

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For a PN-polynomial f, we have

$$f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \iff f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+$$

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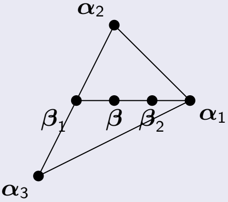
$$f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \iff f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}_+^n$$

Hence to represent a SONC PN-polynomial as a sum of binomial squares, we do not require the denominators of coordinates of points in $\mathscr{A}$-rational mediated sets to be odd. This enables us to decrease the number of binomial squares.

PN-polynomials

An example

Let $f = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ be Motzkin's polynomial and $\mathscr{A} = \{\boldsymbol{\alpha}_1 = (4,2), \boldsymbol{\alpha}_2 = (2,4), \boldsymbol{\alpha}_3 = (0,0)\}$, $\boldsymbol{\beta} = (2,2)$. Then $\boldsymbol{\beta} = \frac{1}{3}\boldsymbol{\alpha}_1 + \frac{1}{3}\boldsymbol{\alpha}_2 + \frac{1}{3}\boldsymbol{\alpha}_3 = \frac{1}{3}\boldsymbol{\alpha}_1 + \frac{2}{3}(\frac{1}{2}\boldsymbol{\alpha}_2 + \frac{1}{2}\boldsymbol{\alpha}_3)$. Let $\boldsymbol{\beta}_1 = \frac{1}{2}\boldsymbol{\alpha}_2 + \frac{1}{2}\boldsymbol{\alpha}_3$ such that $\boldsymbol{\beta} = \frac{1}{3}\boldsymbol{\alpha}_1 + \frac{2}{3}\boldsymbol{\beta}_1$. Let $\boldsymbol{\beta}_2 = \frac{2}{3}\boldsymbol{\alpha}_1 + \frac{1}{3}\boldsymbol{\beta}_1$. It is easy to check that $M = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\}$ is an $\mathscr{A}$-rational mediated set containing $\boldsymbol{\beta}$.



By a simple computation, we have $f = (1 - xy^2)^2 + 2(x^{\frac{1}{2}}y - x^{\frac{3}{2}}y)^2 + (xy - x^2y)^2$. Here we represent f as a sum of three binomial squares with rational exponents.

Converting to PN-polynomials

Let $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ and let
$\tilde{f} = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} |d_{\boldsymbol{\beta}}|\mathbf{x}^{\boldsymbol{\beta}}$ be its associated PN-polynomial.

Fact: $f \in \mathrm{SONC} \iff \tilde{f} \in \mathrm{SONC}.$

Converting to PN-polynomials

Let $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ and let $\tilde{f} = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} |d_{\boldsymbol{\beta}}|\mathbf{x}^{\boldsymbol{\beta}}$ be its associated PN-polynomial.

Fact: $f \in \mathrm{SONC} \iff \tilde{f} \in \mathrm{SONC}$.

Hence we can replace f by $\tilde{f}$ in (SONC) without changing the optimal value:

$$(\text{SONC-PN}): \quad \xi_{sonc} = \begin{cases} \sup & \xi \\ \text{s.t.} & \tilde{f}(\mathbf{x}) - \xi \in \mathrm{SONC}. \end{cases}$$

SONC optimization via second order cone programming

Suppose $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. Let $\{(\mathscr{A}_k, \boldsymbol{\beta}_k)\}_{k=1}^{l}$ be a simplex cover with $\mathscr{A}_k \subseteq \Lambda(f), \forall k$ and $\Gamma(f) \subseteq \cup_{k=1}^{l}\{\boldsymbol{\beta}_k\}$.

For each k, let M_k be an $\mathscr{A}_k$-rational mediated set containing $\boldsymbol{\beta}_k$ and $s_k = \#M_k\backslash\mathscr{A}_k$. For each $\mathbf{u}_i^k \in M_k\backslash\mathscr{A}_k$, let us write $\mathbf{u}_i^k = \frac{1}{2}(\boldsymbol{v}_i^k + \mathbf{w}_i^k)$. Let $\tilde{\mathscr{A}} = \{\boldsymbol{\alpha} \in \Lambda(f) \mid \boldsymbol{\alpha} \notin \cup_{\boldsymbol{\beta}\in\Gamma(f)} \cup_{\Delta\in\mathscr{F}(\boldsymbol{\beta})} V(\Delta)\}$. Then we can relax (SONC-PN) to a second order cone program (SOCP)

$$\xi_{socp} := \begin{cases} \sup & \xi \\ \text{s.t.} & \tilde{f}(\mathbf{x}) - \xi = \sum_{k=1}^{l}\sum_{i=1}^{s_k}(2a_i^k\mathbf{x}^{\boldsymbol{v}_i^k} + b_i^k\mathbf{x}^{\mathbf{w}_i^k} - 2c_i^k\mathbf{x}^{\mathbf{u}_i^k}) + \sum_{\boldsymbol{\alpha}\in\tilde{\mathscr{A}}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}, \\ & (a_i^k, b_i^k, c_i^k) \in \mathbf{K}, \quad \forall i, k, \end{cases}$$

where $\mathbf{K}$ be a 3-dimensional second order cone.

Note: $\xi_{socp} \le \xi_{sonc} \le \xi^*$

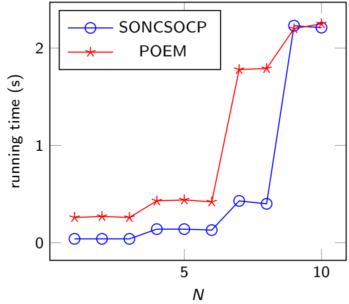
Numerical experiments

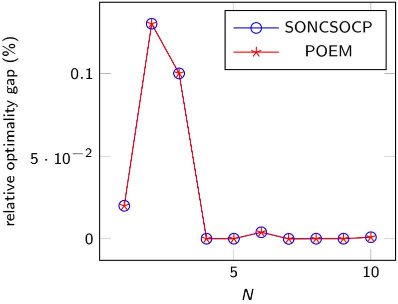
Experiment settings

- SONCSOCP: our tool for SONC optimization via SOCP with Mosek as a SOCP solver
- POEM: Seidler and Wolff's tool for SONC optimization with ECOS as a geometric programming solver
- Benchmarks: Random polynomials generated by Seidler and Wolff
- Relative optimality gap: $\frac{|\xi_{min}-\xi_{lb}|}{|\xi_{min}|}$, where ξ_{min} is a local minimum provided by a local solver and ξ_{lb} is the optimal value given by SONCSOCP or POEM

Results for random polynomials with standard simplex Newton polytopes

Take $N = 10$ polynomials
Number of variables: $10 \sim 40$, degree: $40 \sim 60$, number of terms: $20 \sim 100$

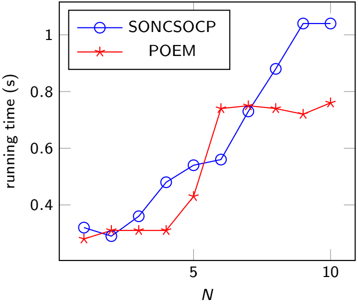


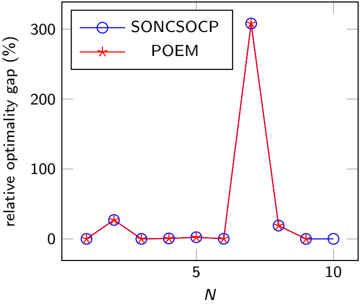


Results for random polynomials with general simplex Newton polytopes

Take $N = 10$ polynomials

Number of variables: 10, degree: $20 \sim 60$, number of terms: $20 \sim 30$





Results for random polynomials with arbitrary Newton polytopes

Take $N = 20$ polynomials

Number of variables: 10, degree: $20 \sim 50$, number of terms: $30 \sim 300$