Exploiting Term Sparsity in SOS Decompositions

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Certifying nonnegativity of polynomials

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- Certifying nonnegativity of multivariate polynomials is a central problem in real algebraic geometry which has applications in polynomial optimization and many other fields such as control, engineering, probability, combinatorics, and physics.
- Generally, this is an NP-hard problem.

Nonnegative polynomials and polynomial optimization

The unconstrained polynomial optimization problem can be formulated as follows:

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It is equivalent to

$$\begin{cases}\text{maximize} & \xi\\ \text{subject to} & f(\mathbf{x})-\xi\geq 0.\end{cases}$$

Sums of squares

A classical approach for certifying nonnegativity of polynomials is the use of sums of squares.

Sums of squares

Given a polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$, if there exist polynomials $f_1, \ldots, f_m \in \mathbb{R}[\mathbf{x}]$ such that

$$f = \sum_{i=1}^{m} f_i^2,$$

then we say that f is a sum of squares (SOS).

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Theorem (Hilbert)

In the univariate case, the quadratic case and the bivariate quartic case, every nonnegative polynomial admits an SOS decomposition.

Except these three cases, there are nonnegative polynomials which cannot decompose into an SOS.
(Motzkin's polynomial: $x^4y^2 + x^2y^4 + 1 - 3x^2y^2$)

SOS decompositions

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- M: a monomial basis
  f admits an SOS decomposition
  $\Longleftrightarrow \exists$ a positive semidefinite matrix Q s.t. $f = M^T QM$
  $\rightsquigarrow$ effectively solved by semidefinite programming (SDP)
  (Parrilo 2000, Lasserre 2001)

Exploiting Sparsity

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- The size of the corresponding semidefinite program problem grows rapidly as the size of the polynomial increases.
- To deal with large polynomials, sparsity must be exploited.
- Newton polytopes (Reznick, 1978), correlative sparsity patterns (Waki et al., 2006), sign-symmetries (Löfberg, 2009), facial reduction (Permenter and Parrilo, 2014), minimal coordinate projections (Permenter and Parrilo, 2015).

Cross sparsity pattern

In order to exploit term sparsity of polynomials, we introduce the notion of cross sparsity patterns. Suppose $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with $\mathrm{supp}(f) = \mathscr{A} \subseteq \mathbb{N}^n$ and fix a monomial basis $\mathbf{x}^{\mathscr{B}} = \{\mathbf{x}^{\boldsymbol{\omega}_1}, \ldots, \mathbf{x}^{\boldsymbol{\omega}_r}\}$ with $\mathscr{B} = \{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r\}$. The cross sparsity pattern of f is represented by an $r \times r$ $(0,1)$-matrix $\mathbf{R}_{\mathscr{A}} = (R_{ij})$ whose elements are given by

$$R_{ij} = \begin{cases} 1, & \boldsymbol{\omega}_i + \boldsymbol{\omega}_j \in \mathscr{A}, \\ 0, & \text{otherwise.} \end{cases}$$

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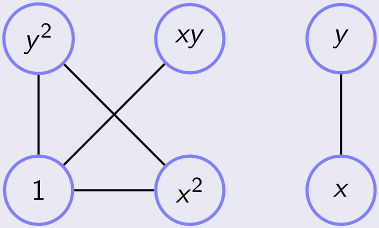
Given a cross sparsity pattern matrix $\mathbf{R}_{\mathscr{A}} = (R_{ij})$, the adjacency graph $G(V_{\mathscr{A}}, E_{\mathscr{A}})$ with $V_{\mathscr{A}} = \{1, 2, \ldots, r\}$ and $E_{\mathscr{A}} = \{\{i,j\} \mid i,j \in V_{\mathscr{A}}, i < j, R_{ij} = 1\}$ is called the cross sparsity pattern graph.

Cross sparsity pattern graph

Example

Let $f = x^2y^2 + x^2 + y^2 + 1 - xy$ and a monomial basis for f is $\{1, x, y, xy, x^2, y^2\}$. Then the cross sparsity pattern graph of f is



Chordal graph

Definition

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A graph is called a chordal graph if all its cycles of length at least four have a chord.

- A chord in a cycle $\{v_1, v_2, \ldots, v_k\}$ is an edge (v_i, v_j) that joins two nonconsecutive nodes in the cycle.
- Any non-chordal graph $G(V, E)$ can always be extended to a chordal graph $\widetilde{G}(V, \widetilde{E})$ by adding appropriate edges to E, which is called a chordal extension of $G(V, E)$.

Chordal graph

Let $G(V, E)$ be a graph.

- A clique $C \subseteq V$ of G is a subset of nodes who induces a complete subgraph.
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It is known that maximal cliques of a chordal graph can be enumerated efficiently in linear time in the number of nodes and edges of the graph.

Sparse PSD matrix

Given a graph $G(V, E)$, let $E^{\star} := E \cup \{(i, i) \mid i \in V\}$. Define

$$S^{r}(E, 0) := \{X \in S^{r} \mid X_{ij} = X_{ji} = 0 \text{ if } (i, j) \notin E^{\star}\}$$

and

$$S_{+}^{r}(E, 0) := \{X \in S^{r}(E, 0) \mid X \succeq 0\}.$$

Sparse SOS relaxation

Suppose $\mathscr{A} \subseteq \mathbb{N}^n$ and $\mathbf{x}^{\mathscr{B}} = \{\mathbf{x}^{\boldsymbol{\omega}_1}, \ldots, \mathbf{x}^{\boldsymbol{\omega}_r}\}$ is a monomial basis. Let the set of SOS polynomials supported on $\mathscr{A}$ be

$$\Sigma(\mathscr{A}) := \{f \in \mathbb{R}[\mathscr{A}] \mid \exists Q \in S_+^r \text{ s.t. } f = (\mathbf{x}^{\mathscr{B}})^T Q \mathbf{x}^{\mathscr{B}}\}.$$

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Let $G(V_{\mathscr{A}}, E_{\mathscr{A}})$ be the cross sparsity pattern graph and $\widetilde{G}(V_{\mathscr{A}}, \widetilde{E}_{\mathscr{A}})$ a chordal extension. To maintain the sparsity of f in the Gram matrix Q, we consider a subset of the SOS polynomials supported on $\mathscr{A}$

$$\widetilde{\Sigma}(\mathscr{A}) := \{f \in \mathbb{R}[\mathscr{A}] \mid \exists Q \in S_+^r(\widetilde{E}_{\mathscr{A}}, 0) \text{ s.t. } f = (\mathbf{x}^{\mathscr{B}})^T Q \mathbf{x}^{\mathscr{B}}\}.$$

Blocking SOS decomposition

Theorem (Wang, Li and Xia, 2019)

Given $\mathscr{A} \subseteq \mathbb{N}^n$, assume that $\mathbf{x}^{\mathscr{B}} = \{\mathbf{x}^{\boldsymbol{\omega}_1}, \ldots, \mathbf{x}^{\boldsymbol{\omega}_r}\}$ is a monomial basis and a chordal extension of the cross sparsity pattern graph is $\widetilde{G}(V_{\mathscr{A}}, \widetilde{E}_{\mathscr{A}})$. Let $C_1, C_2, \ldots, C_t \subseteq V_{\mathscr{A}}$ denote the maximal cliques of $\widetilde{G}(V_{\mathscr{A}}, \widetilde{E}_{\mathscr{A}})$ and $\mathscr{B}_k = \{\boldsymbol{\omega}_i \in \mathscr{B} \mid i \in C_k\}$, $k = 1, 2, \ldots, t$. Then, $f \in \widetilde{\Sigma}(\mathscr{A})$ if and only if there exist $f_k \in \mathbb{R}[\mathscr{B}_k]^2 := \{\sum g_j^2 \mid \mathrm{supp}(g_j) = \mathscr{B}_k\}$, $k = 1, \ldots, t$ such that

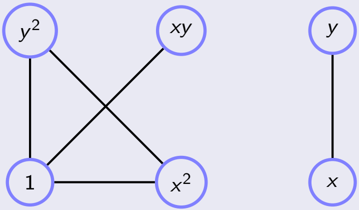
$$f = \sum_{k=1}^{t} f_k.$$

Comparison with correlative sparsity patterns

Correlative sparsity patterns focus on the sparsity of variables, while cross sparsity patterns focus on the sparsity of terms.

Example

Consider the polynomial $f = x^2y^2 + x^2 + y^2 + 1 - xy$. A monomial basis for f is $\{1, x, y, xy, x^2, y^2\}$. The correlative sparsity pattern graph of f is a complete graph, and hence the corresponding Gram matrix of f cannot be blocked. On the other hand, the cross sparsity pattern graph of f has three maximal cliques, corresponding to $\{1, x^2, y^2\}$, $\{1, xy\}$ and $\{x, y\}$ respectively. Hence, the corresponding Gram matrix of f can be blocked into one 3×3 submatrix and two 2×2 submatrices.



Numerical Experiments

Example 1

Let $B_m = (\sum_{i=1}^{3m+2} x_i^2)((\sum_{i=1}^{3m+2} x_i^2)^2 - 2\sum_{i=1}^{3m+2} x_i^2 \sum_{j=1}^{m} x_{i+3j+1}^2)$, where $x_{3m+2+r} = x_r$. For any $m \in \mathbb{N}^*$, B_m is homogeneous and is an SOS polynomial.

Table: Results for B_m

		SparseSOS		Yalmip		SOSTOOLS		SparsePOP	
m	#supp	#block	time	#block	time	#block	time	#block	time
1	35	$5\times5, 10\times1$	0.01s	$5\times5, 10\times1$	0.45s	1×35	0.95s	1×56	0.54s
2	104	$8\times8, 56\times1$	0.04s	$8\times8, 56\times1$	0.95s	1×120	2.59s	1×165	4.66s
3	242	$11\times11, 165\times1$	0.15s	$11\times11, 165\times1$	1.18s	1×286	34.00s	1×364	93.9s
4	476	$14\times14, 364\times1$	0.45s	$14\times14, 364\times1$	2.94s	1×560	423s	1×680	764s
5	833	$17\times17, 680\times1$	1.56s	1×969	OM	1×969	OM	OM	
10	5408	$32\times32, 4960\times1$	65.55s						

Example 2

Randomly generated sparse polynomials.

Table: Results for randomly generated polynomials

				SparseSOS		Yalmip	
	#var	deg	#supp	#block	time	#block	time
F_1	10	12	590	$187, 5, 6 \times 2, 44 \times 1$	179.2s	2×248	315.60s
F_2	10	12	310	$83, 3, 4 \times 2, 37 \times 1$	4.42s	131	16.34s
F_3	10	12	504	$162, 6, 4, 6 \times 2, 34 \times 1$	63.86s	218	116.09s
F_4	10	12	873	$303, 8, 3 \times 2, 40 \times 1$	1850.54s	357	OM
F_5	10	12	709	$238, 4, 4 \times 3, 12, 55 \times 1$	633.51s	331	OM
F_6	10	12	927	$231, 3, 2 \times 2, 23 \times 1$	470.40s	261	297.40s
F_7	10	16	306	$110, 10, 6, 3 \times 4, 5 \times 3, 22 \times 2, 192 \times 1$	29.95s	389	OM
F_8	10	16	255	$62, 8, 5, 4, 2 \times 3, 2 \times 2, 131 \times 1$	32.09s	220	185.35s
F_9	10	16	228	$56, 13, 2 \times 6, 2 \times 4, 4 \times 3, 12 \times 2, 107 \times 1$	11.24s	232	200.43s
F_{10}	10	20	1344	$4658, 7, 2 \times 5, 3 \times 4, 7 \times 3, 16 \times 2, 29 \times 1$	OM	4769	OM
F_{11}	10	20	1392	$5012, 5, 3, 3 \times 2, 20 \times 1$	OM	5046	OM
F_{12}	10	20	1845	$4528, 7, 3, 5 \times 2, 28 \times 1$	OM	4576	OM

Thank you!