Exploiting Sparsity in Large-Scale Polynomial Optimization

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Polynomial optimization problem

The polynomial optimization problem (POP):

$$
(\mathrm{Q}): \quad \begin{array}{rll} f^* := & \inf & f \\ & \mathrm{s.t.} & g_j \geq 0, \quad j = 1, \ldots, m, \\ & & (h_i = 0, \quad i = 1, \ldots, m') \end{array}
$$

where $f, g_j \, (h_i) \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$.

In general, the problem (Q) is non-convex, NP-hard.

Applications

- Discrete optimization (e.g. the max-cut problem)
- Optimal power flow
- Truncated least squares (TLS) estimation
- Sparse polynomial interpolation
- Optimal control
- Deep learning
- Quantum information
- ......

What do we care about?

- Compute the global optimal value
- Extract global minimizers
- Certify global optimality
- Approximate the global optimal value if the exact computation is impossible/unnecessary/expensive

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The moment-SOS hierarchy (also known as Lasserre's hierarchy) is a powerful tool to handle POPs and to answer all these questions.

What does "moment" mean?

Assume $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \ldots, m\}$. The POP (Q) is equivalent to

$$\inf_{\mu \in \mathcal{M}(\mathbf{K})_+} \{\int_{\mathbf{K}} f(\mathbf{x}) \,\mathrm{d}\mu : \mu(\mathbf{K}) = 1\}. \tag{1}$$

Let $y_{\boldsymbol{\alpha}} = \int_{\mathbf{K}} \mathbf{x}^{\boldsymbol{\alpha}} \,\mathrm{d}\mu$ (moment) for $\boldsymbol{\alpha} \in \mathbb{N}^n$. Then (1) can be rewritten as

$$\inf_{\mathbf{y}} \{L_{\mathbf{y}}(f) = \sum_{\boldsymbol{\alpha} \in \mathrm{supp}(f)} f_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} : \exists \mu \in \mathcal{M}(\mathbf{K})_+ \text{ s.t. } \mathbf{y} \sim \mu \text{ and } y_{\mathbf{0}} = 1\}. \tag{2}$$

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Question: Which sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ admits a finite Borel measure representation with support contained in $\mathbf{K}$?

What does “moment” mean?

The d-order moment matrix $M_d(\mathbf{y})$ is defined by $[M_d(\mathbf{y})]_{\boldsymbol{\beta}\boldsymbol{\gamma}} = y_{\boldsymbol{\beta}+\boldsymbol{\gamma}}$ for $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n_d$.
Given $g \in \mathbb{R}[\mathbf{x}]$, the d-order localizing matrix $M_{d-d_j}(g\mathbf{y})$ is defined by $[M_{d-d_j}(g\mathbf{y})]_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \sum_{\boldsymbol{\alpha}\in\mathrm{supp}(g)} g_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}}$ for $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n_{d-d_j}$ ($d_j = \lceil \deg(g_j)/2 \rceil$).

Theorem

Assume Archimedean's condition holds. The sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ has a finite Borel representing measure with support contained in K if and only if $M_d(\mathbf{y}) \succeq 0$, $M_{d-d_j}(g_j\mathbf{y}) \succeq 0$ for all j and d.

What does "moment" mean?

By truncating the order of moments, we then obtain a series of moment relaxations (indexed by d) of (Q) to approximate f^* from below:

$$(\mathrm{Q}_d): \quad \begin{array}{rll} \theta_d := & \inf & L_{\mathbf{y}}(f) \\ & \text{s.t.} & M_d(\mathbf{y}) \succeq 0, \\ & & M_{d-d_j}(g_j\mathbf{y}) \succeq 0, \quad j = 1, \ldots, m, \\ & & y_{\mathbf{0}} = 1. \end{array}$$

Here, $d_0 = 0, d_j = \lceil \deg(g_j)/2 \rceil$.

This is actually a semidefinite programming (SDP) problem, effectively solved by interior-point solvers (e.g. MOSEK) or first-order solvers (e.g. SDPNAL).

What does “SOS” mean?

Assume $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, j = 1, \ldots, m\}$. The dual of (Q) reads as

$$f^* = \sup_{\lambda}\{\lambda : f(\mathbf{x}) - \lambda \geq 0 \text{ over } \mathbf{K}\}. \tag{3}$$

The convex cone $P_{\mathbf{K}}(\mathbf{x}) := \{g(\mathbf{x}) \mid g(\mathbf{x}) \geq 0 \text{ over } \mathbf{K}\}$ is intractable!

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Question: How to effectively approximate $P_{\mathbf{K}}(\mathbf{x})$ by tractable subsets (or supsets)?

What does "SOS" mean?

$\Sigma(\mathbf{x}) := \{f \in \mathbb{R}[\mathbf{x}] \mid f = \sum_i f_i^2, f_i \in \mathbb{R}[\mathbf{x}]\}$ (SOS polynomials)
Given $\mathbf{g} = \{g_j\}_{j=1}^m \subseteq \mathbb{R}[\mathbf{x}]$, the quadratic module generated by $\mathbf{g}$ is
$\mathcal{Q}_{\mathbf{g}} := \{\sigma_0 + \sum_{j=1}^m \sigma_j g_j \mid \sigma_j \in \Sigma(\mathbf{x}), j = 0, 1, \ldots, m\}$,
and the truncated quadratic module of degree $2d$ is (with $g_0 := 1$)
$\mathcal{Q}_{\mathbf{g},2d} := \{\sigma_0 + \sum_{j=1}^m \sigma_j g_j \mid \sigma_j \in \Sigma(\mathbf{x}), \deg(\sigma_j g_j) \le 2d, j = 0, 1, \ldots, m\}$.

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Theorem (Putinar's Positivstellensatz)

Assume Archimedean's condition holds. If $f > 0$ over $\mathbf{K}$, then $f \in \mathcal{Q}_{\mathbf{g}}$.

Replacing $P_{\mathbf{K}}(\mathbf{x})$ by $\mathcal{Q}_{\mathbf{g},2d}$, we then obtain a series of SOS relaxations (indexed by d) of (Q) to approximate f^* from below:

$$(\mathrm{Q}_d)^*: \quad \theta_d^* := \begin{array}{ll} \sup & \lambda \\ \text{s.t.} & f - \lambda \in \mathcal{Q}_{\mathbf{g},2d}. \end{array}$$

This is actually the dual SDP problem of the moment relaxation.

The moment-SOS hierarchy

$$\begin{array}{ccccc} & & f^* & & \\ & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\leq$} & \\ & \vdots & & \vdots & \\ & \geq & & \geq & \\ \text{(the moment relaxation)} & \theta_d & \text{“}=\text{”} & \theta_d^* & \text{(the SOS relaxation)} \\ & \geq & & \geq & \\ & \vdots & & \vdots & \\ & \geq & & \geq & \\ & \theta_{\underline{d}} & \text{“}=\text{”} & \theta_{\underline{d}}^* & \end{array}$$

$$\underline{d} := \max\{\deg(f)/2, d_1, \ldots, d_m\}$$

Asymptotical convergence and finite Convergence

Under Archimedean's condition ($\approx$ compactness): there exists $N > 0$ s.t. $N - ||\mathbf{x}||^2 \in \mathcal{Q}_{\mathbf{g}}$, we have

- $\theta_d \uparrow f^*$ and $\theta_d^* \uparrow f^*$ as $d \to \infty$ (Lassere, 2001);
- Finite convergence happens generically (Nie, 2014);
- We can verify global optimality by the so-called rank condition (flat extension/truncation);
- We can easily extract minimizers when the rank condition is satisfied.

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Important Message: The moment-SOS hierarchy enable us to approximate/retrieve the global optimum/optimizers via solving a sequence of SDPs with increasing sizes.

Scalability

The size of SDP (considering $(\mathrm{Q}_d)^*$) at relaxation order d:

- SDP matrix: $\binom{n+d}{d}$
- #equality constraint: $\binom{n+2d}{2d}$

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Exploiting structure:

- quotient ring
- symmetry
- constant trace property
- sparsity (correlative sparsity and term sparsity)

Correlative sparsity (Waki et al., 2006)

The basic idea is to partition the variables into cliques according to the correlations between variables.

Correlative sparsity pattern (csp) graph $G^{\mathrm{csp}}(V, E)$:
$V := \{x_1, \ldots, x_n\}$
$\{x_i, x_j\} \in E \Longleftrightarrow x_i, x_j$ appear in the same term of f or appear in the same constraint g_j

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We then construct moment/localizing matrices with respect to the variables involved in each maximal clique of the csp graph:

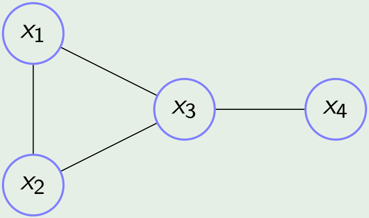
$$I_k \longmapsto M_d(\mathbf{y}, I_k), M_{d-d_j}(g_j\mathbf{y}, I_k)$$

Correlative sparsity

Example

Consider $f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$ and $g_1 = 1 - x_1^2 - x_2^2 - x_3^2$, $g_2 = 1 - x_3x_4$.

Figure: The csp graph for f and $\{g_1, g_2\}$



Two maximal cliques: $\{x_1, x_2, x_3\}$ and $\{x_3, x_4\}$

The correlative sparsity adapted moment-SOS hierarchy

- If the csp graph is chordal (otherwise we need a chordal extension), then the correlative sparsity adapted moment-SOS hierarchy shares the same convergence as the standard one;
- We can still verify global optimality by the (adapted) rank condition;
- We can still extract global minimizers if certain rank conditions are satisfied;
- Significantly improve scalability if the sizes of maximal cliques of the csp graph are small (e.g. ≤ 10).

Term sparsity

In contrast with correlative sparsity concerning variables, term sparsity treats sparsity at the term level.

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$V_d(\mathbf{x}) := \{1, x_1, \ldots, x_n, x_1^d, \ldots, x_n^d\}$ the monomial basis of degree $\leq d$.

Term sparsity pattern (tsp) graph $G^{\text{tsp}}(V, E)$ (with relaxation order d):
$V := V_d(\mathbf{x})$
$\{\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\} \in E \iff \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} = \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}} \in \operatorname{supp}(f) \cup \bigcup_{j=1}^{m} \operatorname{supp}(g_j) \cup V_d(\mathbf{x})^2$

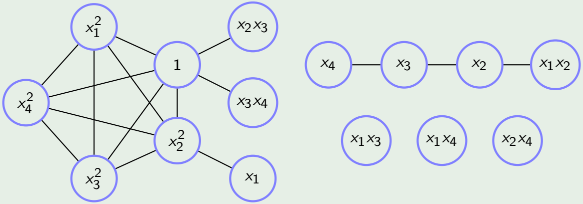
(For $f = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$, $\operatorname{supp}(f) := \{\mathbf{x}^{\boldsymbol{\alpha}} \mid f_{\boldsymbol{\alpha}} \neq 0\}$)

Term sparsity

Example

Consider $f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$ and $g_1 = 1 - x_1^2 - x_2^2 - x_3^2$, $g_2 = 1 - x_3x_4$.

Figure: The tsp graph for f and $\{g_1, g_2\}$ with $d = 2$



Term sparsity

Suppose the tsp graph $G^{\rm tsp}$ has connected components: $\mathscr{B}_1, \ldots, \mathscr{B}_t$. So

$$V_d(\mathbf{x}) = \bigsqcup_{i=1}^{t} \mathscr{B}_i.$$

For each $\mathscr{B}_i$, we construct a block of the moment matrix: $M_{\mathscr{B}_i}(\mathbf{y})$.

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In such a way, we replace one big matrix $M_d(\mathbf{y})$ by a series of smaller matrices $M_{\mathscr{B}_i}(\mathbf{y}), i = 1, \ldots, t$ in the moment relaxation.

Remark: The same thing can be also done for the localizing matrices $M_{d-d_j}(\mathbf{y}), j = 1, \ldots, m$.

Extending to an iterative procedure

For simplicity, we consider the unconstrained case. For a graph $G(V, E)$ with nodes $V_d(\mathbf{x})$ ($d = \deg(f)/2$), define

$$\mathrm{supp}(G) := \{\mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \mid \{\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\} \in E\}.$$

Let $G^{(0)} = G^{\mathrm{tsp}}$. We iteratively define a sequence of graphs $(G^{(k)})_{k\geq 1}$ via two successive operations:

1. Support extension: let $F^{(k)}$ be the graph with nodes $V_d(\mathbf{x})$ and edges

$$E(F^{(k)}) := \{\{\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}}\} \mid \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in \mathrm{supp}(G^{(k-1)}) \cup V_d(\mathbf{x})^2\}$$

2. Block closure: $G^{(k)} = \overline{F^{(k)}}$, i.e. $G^{(k)}$ is obtained by completing every connected components of $F^{(k)}$

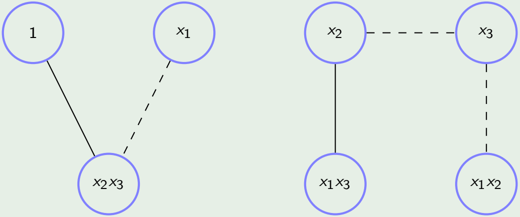
Support extension

Example

Consider the following graph $G(V, E)$ with

$$V = \{1, x_1, x_2, x_3, x_2x_3, x_1x_3, x_1x_2\} \text{ and } E = \{\{1, x_2x_3\}, \{x_2, x_1x_3\}\}.$$

Figure: The support extension of G



Block closure

Example

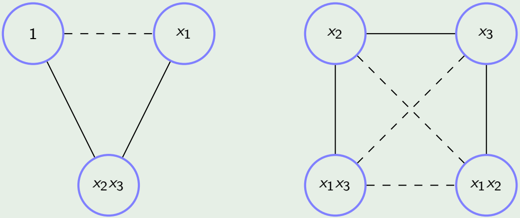
Consider the following graph $G(V,E)$ with

$$V = \{1, x_1, x_2, x_3, x_2x_3, x_1x_3, x_1x_2\}$$

and

$$E = \{\{1, x_2x_3\}, \{x_2, x_1x_3\}, \{x_1, x_2x_3\}, \{x_2, x_3\}, \{x_3, x_1x_2\}\}.$$

Figure: The block closure of G



The term sparsity adapted moment-SOS (TSSOS) hierarchy

Let $\mathscr{B}_1^{(k)}, \ldots, \mathscr{B}_{t_k}^{(k)}$ be the connected components of $G^{(k)}$. For each $k \geq 1$, let us consider

$$(\mathrm{Q}^k): \quad \begin{array}{rll} \theta^{(k)} := & \inf & L_{\mathbf{y}}(f) \\ & \text{s.t.} & M_{\mathscr{B}_i^{(k)}}(\mathbf{y}) \succeq 0, \quad i = 1, \ldots, t_k \\ & & y_{\mathbf{0}} = 1. \end{array}$$

One then obtains

$$\theta_{\mathrm{sdsos}} \leq \theta^{(1)} \leq \theta^{(2)} \leq \cdots \leq f^*.$$

We call $(\mathrm{Q}^k), k = 1, 2, \ldots$ the TSSOS hierarchy for (Q) and k the sparse order.

A two-level hierarchy of lower bounds

The above procedure can be extended to the constrained case. As a consequence, we obtain a two-level hierarchy of lower bounds for f^*: $(\underline{d} := \max\{\deg(f)/2, d_1, \ldots, d_m\})$

$$
\begin{array}{ccccccc}
\theta_{\underline{d}}^{(1)} & \leq & \theta_{\underline{d}}^{(2)} & \leq & \cdots & \leq & \theta_{\underline{d}} \\
\wedge| & & \wedge| & & & & \wedge| \\
\theta_{\underline{d}+1}^{(1)} & \leq & \theta_{\underline{d}+1}^{(2)} & \leq & \cdots & \leq & \theta_{\underline{d}+1} \\
\wedge| & & \wedge| & & & & \wedge| \\
\vdots & & \vdots & & \vdots & & \vdots \\
\wedge| & & \wedge| & & & & \wedge| \\
\theta_{d}^{(1)} & \leq & \theta_{d}^{(2)} & \leq & \cdots & \leq & \theta_{d} \\
\wedge| & & \wedge| & & & & \wedge| \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

The TSSOS hierarchy

Regarding the TSSOS hierarchy, we have

- For QCQP, $\theta_1^{(1)} = \theta_{\mathrm{shor}}$;
- Fixing a sparse order k, the sequence $(\theta_d^{(k)})_{d\geq\underline{d}}$ is monotone nondecreasing;
- Fixing a relaxation order d, the sequence $(\theta_d^{(k)})_{k\geq1}$ converges to θ_d in finitely many steps.

The connection to sign symmetries

Definition

Given a finite set $\mathscr{A} \subseteq \mathbb{N}^n$, the sign symmetries of $\mathscr{A}$ are defined by all vectors $\mathbf{r} \in \mathbb{Z}_2^n$ such that $\mathbf{r}^T \boldsymbol{\alpha} \equiv 0 \pmod 2$ for all $\boldsymbol{\alpha} \in \mathscr{A}$.

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Example

Let $\mathscr{A} = \{\begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\}$. The sign symmetries of $\mathscr{A}$ consist of $\mathbf{r}_1 = \begin{bmatrix}0\\0\end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

The connection to sign symmetries

Let $\mathscr{A} = \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid \mathbf{x}^{\boldsymbol{\alpha}} \in \mathrm{supp}(f) \cup \bigcup_{j=1}^{m} \mathrm{supp}(g_j)\}$.
The sign symmetries $R = [\mathbf{r}_1, \ldots, \mathbf{r}_s]$ of $\mathscr{A}$ induces a partition of the bases $V_d(\mathbf{x})$, $V_{d-d_j}(\mathbf{x})$, $j = 1, \ldots, m$:

$$\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}} \text{ belong to the same block} \iff R^T(\boldsymbol{\alpha} + \boldsymbol{\beta}) \equiv 0 \pmod 2.$$

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$$\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}} \text{ belong to the same block} \iff R^T(\boldsymbol{\alpha} + \boldsymbol{\beta}) \equiv 0 \pmod{2}.$$

Theorem (Wang, Magron and Lasserre (2020))

Fixing a relaxation order d, the partition of monomial bases $V_d(\mathbf{x})$, $V_{d-d_j}(\mathbf{x})$ at the final step of the TSSOS hierarchy is the one induced by the sign symmetries of the above $\mathscr{A}$.

A sparse representation theorem

Theorem (Wang, Magron and Lasserre (2020))

Assume that the quadratic module $\mathcal{Q}_{\mathbf{g}}$ is Archimedean and that f is positive on $\mathbf{K}$. Let $\mathscr{A} = \mathrm{supp}(f) \cup \bigcup_{j=1}^{m} \mathrm{supp}(g_j)$ and R be the sign symmetries of $\mathscr{A}$. Then f can be represented as

$$f = \sigma_0 + \sum_{j=1}^{m} \sigma_j g_j,$$

for some SOS polynomials $\sigma_0, \sigma_1, \ldots, \sigma_m$ satisfying $R^T \boldsymbol{\alpha} \equiv 0 \pmod 2$ for any $\mathbf{x}^{\boldsymbol{\alpha}} \in \mathrm{supp}(\sigma_j), j = 0, \ldots, m$.

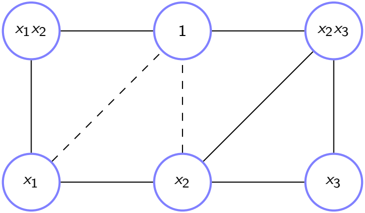
Extensions

- Replacing block closure by chordal extension;
- Exploiting correlative sparsity and term sparsity simultaneously;
- Exploiting quotient structure and term sparsity simultaneously;
- Extending to complex polynomial optimization ($\mathbb{C}[\mathbf{x}]$);
- Extending to noncommutative polynomial optimization ($\mathbb{R}\langle Y\rangle$);
- Extending to polynomial optimization over $\mathbb{C}\langle Y\rangle$, $\mathbb{R}[\mathbf{x}]\langle Y\rangle$, $\mathbb{C}[\mathbf{x}]\langle Y\rangle$.

Replacing block closure by chordal extension

Let $f = x_1^2 - 2x_1x_2 + x_2^2 - 2x_1^2x_2 + x_1^2x_2^2 - 2x_2x_3 + x_3^2 + x_2^2x_3 - x_2x_3^2 + x_2^2x_3^2$.
A monomial basis: $\{1, x_1, x_2, x_3, x_1x_2, x_2x_3\}$.

Figure: Block closure versus chordal extension



Combining correlative sparsity with term sparsity

The combination of correlative sparsity with term sparsity splits into two steps:

1. Partitioning the variables with respect to the maximal cliques of the csp graph;
2. For each subsystem involving variables from one maximal clique, applying the above iterative procedure to exploit term sparsity.

In doing so, we again obtain a two-level hierarchy of lower bounds for f^*, which is called the CS-TSSOS hierarchy.

Softwares

All sparsity-exploiting techniques (reduced monomial basis, quotient structure, correlative sparsity, term sparsity, combined correlative-term sparsity) have been implemented in the following two softwares (freely available on GitHub):

- `TSSOS`: solving commutative polynomial optimization
- `NCTSSOS`: solving noncommutative polynomial optimization

Besides, as an application to a specific SOS program

- `SparseJSR`: computing joint spetral radius for a tuple of matrices

Randomly generated polynomials of the SOS form

TSSOS, GloptiPoly, Yalmip: MOSEK   SparsePOP: SDPT3

Table: Running time (in seconds) comparison with GloptiPoly, Yalmip and SparsePOP for minimizing randomly generated sparse polynomials of the SOS form with the same optimum; the symbol "-" indicates out of memory

n	$2d$	TSSOS	GloptiPoly	Yalmip	SparsePOP
8	8	0.24	306	10	24
8	8	0.34	348	13	130
8	8	0.36	326	19	175
8	10	0.58	-	92	323
8	10	0.53	-	72	1526
8	10	0.38	-	22	134
9	10	0.50	-	44	324
9	10	0.72	-	143	-
9	10	0.79	-	109	284
10	12	2.2	-	474	-
10	12	1.6	-	147	318
10	12	1.8	-	350	404
10	16	15	-	-	-
10	16	14	-	-	-
10	16	12	-	-	-
12	12	8.4	-	-	-
12	12	5.7	-	-	-
12	12	7.4	-	-	-

Randomly generated polynomials with simplex Newton polytopes

Table: Running time (in seconds) comparison with GloptiPoly, Yalmip and SparsePOP for minimizing randomly generated sparse polynomials with simplex Newton polytopes with the same optimum; the symbol "-" indicates out of memory

n	$2d$	TSSOS	GloptiPoly	Yalmip	SparsePOP
8	8	0.36	346	31	271
8	8	0.51	447	24	496
8	8	0.31	257	6.0	178
9	8	1.0	-	-	-
9	8	0.63	-	363	611
9	8	0.76	-	141	578
9	10	6.6	-	322	-
9	10	5.0	-	233	-
9	10	4.9	-	249	-
10	8	1.2	-	-	-
10	8	8.0	-	536	-
10	8	1.0	-	-	-
11	8	1.7	-	655	398
11	8	1.8	-	-	221
11	8	1.9	-	340	293
12	8	10	-	-	-
12	8	7.4	-	-	-
12	8	2.9	-	-	-

AC-OPF problems

Table: The results for AC-OPF problems; the symbol "-" indicates out of memory

n	m	CS ($d = 2$)				CS+TS ($d = 2$)			
		mb	opt	time (s)	rel. gap	mb	opt	time (s)	rel. gap
12	28	28	1.1242e4	0.21	0.00%	22	1.1242e4	0.09	0.00%
20	55	28	1.7543e4	0.56	0.05%	22	1.7543e4	0.30	0.05%
114	315	66	1.3442e5	5.59	0.39%	31	1.3396e5	2.01	0.73%
114	315	120	7.6943e4	94.9	0.00%	39	7.6942e4	14.8	0.00%
72	297	45	4.9927e3	4.43	0.07%	22	4.9920e3	2.69	0.08%
344	971	153	4.2246e5	758	0.06%	44	4.2072e5	96.0	0.48%
344	971	153	2.2775e5	504	0.00%	44	2.2766e5	71.5	0.04%
344	1325	253	–	–	–	31	2.4180e5	82.7	0.11%
344	1325	253	–	–	–	73	1.0470e5	169	0.50%
348	1809	253	–	–	–	34	1.0802e5	278	0.05%
348	1809	253	–	–	–	34	1.2096e5	201	0.03%
766	3322	153	3.3072e6	585	0.68%	44	3.3042e6	33.9	0.77%
1112	4613	231	4.2413e4	3114	0.85%	39	4.2408e4	46.6	0.86%
1112	4613	496	–	–	–	31	7.2396e4	410	0.25%
4356	18257	378	–	–	–	27	1.3953e6	934	0.51%

Eigenvalue minimization for the noncommutative generalized Rosenbrock function

Table: The eigenvalue minimization for the noncommutative generalized Rosenbrock function over $\mathcal{D}$, where $\mathcal{D}$ is defined by $\{1 - X_1^2, \ldots, 1 - X_n^2, X_1 - 1/3, \ldots, X_n - 1/3\}$; the symbol "-" indicates out of memory

n	CS+TS ($d = 2$)			Dense ($d = 2$)		
	mb	opt	time (s)	mb	opt	time (s)
20	3	1.0000	0.14	-	-	-
40	3	1.0000	0.22	-	-	-
60	3	0.9999	0.28	-	-	-
80	3	0.9999	0.35	-	-	-
100	3	0.9999	0.46	-	-	-
200	3	0.9999	0.89	-	-	-
400	3	1.0000	2.40	-	-	-
600	3	1.0000	4.47	-	-	-
800	3	1.0000	6.95	-	-	-
1000	3	0.9999	10.2	-	-	-
2000	3	0.9999	37.2	-	-	-
3000	3	0.9999	87.2	-	-	-
4000	3	0.9998	145	-	-	-

Open questions

- How to certify/prove global optimality in the sparse setting (term sparsity and combined correlative-term sparsity)?
- How to exact global optimizers in the sparse setting (term sparsity and combined correlative-term sparsity)?
- How to choose appropriate chordal extensions for specific applications?
- Is it possible go beyond chordal extension?

What else?

- Finding a smaller monomial basis;
- Relying on other positivity certificates, e.g., Krivine-Stengle's certificate (LP or SDP), the SONC certificate (GP or SOCP), the SAGE certificate (REP);
- Approximating the PSD cone by simple convex cones;
- Developing fast first-order algorithms to solve SDP.

Conclusions and outlooks

- The concept of term sparsity patterns opens a new window to exploit sparsity at the term level for polynomial optimization;
- The TSSOS hierarchy is a powerful tool to handle large-scale polynomial optimization problems;
- One can exploit term sparsity for generalized moment problems (more general than polynomial optimization), SOS programming, SDP problems;
- Fruitful potential applications: optimal power flow, computer vision, control, deep learning, quantum information, tensor decomposition, ...

Main references

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Thanks for your attention!